Universidade Federal do Piauí
Centro de Ciências da Natureza
Pós-Graduação em Matemática
Doutorado em Matemática

Proximal Point Method for Quasi Equilibrium Problems: Exact and Inexact Versions

Edimilson Lopes Dias Júnior

Teresina - 2023

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Tese de Doutorado:

Proximal Point Method for Quasi Equilibrium Problems: Exact and Inexact Versions

Tese de Doutorado submetida a coordenação do Programa de Pós-Graduação em Matemática, da Universidade Federal do Piauí, como requisito parcial para obtenção do grau de Doutor em Matemática. Área de Concentração Otimização.

Orientador:
Prof. Dr. João Carlos de Oliveira Souza

Teresina - 2023

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No dia quinze do mês de setembro de dois mil e vinte e três, às 10:00 horas, na Sala de Seminários do Programa de Pós Graduação em Matemática - PPGMAT/UFPI, reuniram-se os membros da Banca Examinadora composta pelos professores: Dr. **João Carlos de Oliveira Souza** (Presidente), Dr. **Jurandir de Oliveira Lopes** (UFPI), Dr. **Ray Victor Guimarães Serra** (UFPI), Dr. **Pedro Jorge Sousa dos Santos** (UFDPAR), Dr. **Renan Spencer Trindade** (École Polytechnique, França) a fim de julgar a tese do doutorando **Edimilson Lopes Dias Junior**, intitulada *Proximal Point Method For Quasi-Equilibrium Problems: Exact and Inexact Versions* , para obtenção do grau de doutor em Matemática. Aberta a sessão pelo presidente, coube ao candidato na forma regimental, expor o tema de sua tese dentro do tempo regulamentar. Após haver analisado o referido trabalho e arguido o candidato, os membros da Banca Examinadora deliberaram pela **APROVAÇÃO** do mesmo. Nada mais havendo a tratar, foi lavrada a presente ata, que vai assinada pelos membros da Banca Examinadora.

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Declaro que no dia 15 de setembro de 2023, às 10:00 horas participei, de forma remota com os demais membros deste ato público, da banca examinadora de Defesa de Doutorado do discente **Edimilson Lopes Dias Junior** do Programa de Pós-Graduação em Matemática da Universidade Federal do Piauí – UFPI. Considerando o Trabalho avaliado, as arguições de todos os membros da banca e as respostas dadas pelo discente, formalizo para fins de registro, minha decisão de que o discente está **APROVADO**.

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D541p Dias Júnior, dimilson Lopes.
Proximal point method for quasi equilibrium problems: exact and inexact versions / Edimilson Lopes Dias Júnior. -- 2023.
83 f.

Tese (Doutorado) - Universidade Federal do Piauí. Centro de Ciências da Natureza. Programa de Pós-Graduação em Matemática, Teresina, 2023.
"Orientador: Prof. Dr. João Carlos de Oliveira Souza ."

1. Otimização matemática. 2. Quase equilíbrio. 3. Problema de duopólio. 4. Modelo de Cournot. I. Souza, João Carlos de Oliveira. II. Título.

Bibliotecária: Caryne Maria da Silva Gomes - CRB3/1461

Aos meus avós Maria José, João Lopes, Ana Ferreira e Benedito Ferreira (In memoriam).

Agradecimentos

Finalizo aqui a minha jornada acadêmica dentro da instituição que por mais de dez anos foi a minha casa. Quando adentrei na UFPI tinha como pretensões ir longe, o mais longe que eu pudesse e cá estou. Tenho certeza que isso não seria possível sem ajuda de muitas pessoas que tive a honra e a oportunidade de conviver ao longo de todos esses anos e é a elas que me direciono agora. Inicio meus agradecimentos primeiramente agradecendo a Deus por ter me dado saúde e força pra chegar até aqui, não foi uma trajetória fácil e muitas vezes a vontade de desistir existiu em meu coração. Aos meus pais Maria de Fátima e Edimilson Lopes por terem me mostrado o quão importante é a educação e que devemos sempre buscar realizar aquilo que sonhamos não importando o quão difícil isso possa ser. Agradeço a minha esposa Evelyn Larissa que está ao meu lado desde o início dessa jornada no longínquo 2011. De lá pra cá passamos por inúmeras batalhas juntos e sem você certamente a caminhada não teria sido da maneira que foi, obrigado por tudo meu amor(amo você!). Aos amigos, com quem tive a honra de dividir a sala de aula:Alexandre, Antonio Nilson, Atécio, Bruno, Christopher, Dieme, Erisvaldo, Gustavo, Jefferson Brito, João Vinícius, João Santos, José Márcio e Pedro Paulo. Me direciono agora aos professores do departamento de matemática, em especial aos professores do programa de pós-graduação em matemática com quem tive a honra de estudar: Isaias Pereira de Jesus , Halyson Irene Baltazar, Jurandir de Oliveira Lopes , Paulo Sérgio Marques dos Santos e João Xavier da Cruz Neto. Gostaria de fazer um agradecimento especial aos professores Gleison do Nascimento Santos e Rondinele Marcolino Batista, pois ainda no primeiro semestre do doutorado, no ano de 2019 fui acometido por uma infecção grave que quase ceifou minha vida. Agradeço pela compreensão do momento difícil que passei e por terem me ajudado com as disciplinas, ainda que eu não tenha conseguido cursar o semestre por inteiro. Agradeço de forma especial também ao meu orientador, João Carlos de Oliveira Souza por toda a ajuda e ensinamentos ao longo desses anos, o senhor foi fundamental para a realização desse sonho. Agradeço ainda aos membros da minha banca examinadora Ray Guimarães Serra, Jurandir de Oliveira Lopes, Pedro Jorge dos Santos e Renan Spencer Trindade. Por fim, agradeço a CAPES pelo apoio financeiro ao longo desses quatro anos.

" É justo que muito custe, o que muito vale."

Santa Teresa D'Ávila

Resumo

Nesse trabalho propomos duas versões do método ponto proximal para problemas em quase-equilíbrio: Uma versão exata e outra inexata do método. Na versão exata, propomos uma versão com regularização Bregman do método do ponto proximal, de modo que estendemos o trabalho de Burachik e Kassay [4] para o contexto de quase equilíbrio.Enquanto na versão inexata, fazemos a extensão dos resultados obtidos por Santos e Souza em [47], uma vez que utilizamos uma regularização mais geral que a utilizada no respectivo trabalho. Além disso, apresentamos experimentos numéricos afim de ilustrar o desempenho computacional das versões propostas. Por fim, no último capítulo apresentamos uma aplicação de quase equilíbrio para o problema de duopólio através do modelo de Cournot.

Palavras-chave: Método ponto proximal; problemas de quase-equilíbrio; distância de Bregman; problema de duopólio; modelo de Cournot.

Abstract

In this work, we propose two versions of the proximal point method: an exact and an inexact version of the method. In the exact version, we propose a Bregman regularized version of the proximal point method, so we extend the work of Burachik and Cassay [4] to the quasi-equilibrium context. In its inexact version, we extend the results obtained by Santos and Souza in [47]. Furthermore, we present numerical experiments in order to illustrate the computational performance of the proposed versions. Finally, in the last chapter, we present a quasi-equilibrium application for the Cournot duopoly model.

Keywords: Proximal point method; quasi-equilibrium problems; Bregman distance; duopoly problem; Cournot model.

Contents

Introduction

In this work, we study An equilibrium problem, which we will denote by $\mathsf{EP}(f,X)$ consists of finding a point $x^* \in X$ such that

$$f(x^*, y) \geqslant 0, \quad \forall y \in X.$$

The set $X \subset H$ is a non-empty, closed, convex set and H is a Hilbert space.Moreover, $f : X \times X \to \mathbb{R}$ is called equilibrium bifunction. The bifunction f satisfies the following properties:

Assumptions 1.

i) $f(x,x) = 0$ *for all* $x \in X$.

ii) $f(\cdot,\cdot) : X \times X \to \mathbb{R}$ *is jointly weakly continuous (its graph is weakly closed) in the following sense: if* $x, y \in X$ *are such that* $\{x^j\}$ *and* $\{y^j\}$ *weakly converge to* x *and* y *respectively. Then* $f(x^j, y^j)$ *converges in norm to* $f(x,y)$.

iii) $f(x,\cdot)$ *is convex for all* $x \in X$.

iv) f *is monotone, that is,* $f(x,y) + f(y,x) \leqslant 0$ *for all* $x, y \in X$.

v) For any sequence $\{z^n\} \subset X$ *with* $\lim_{n\to+\infty} \|z^n\| = +\infty$*, there exists* $u \in X$ *and* $n_0 \in \mathbb{N}$ *such that* $f(z^n, u) \leqslant 0$ *, for all* $n \geqslant n_0$.

We denote the solutions set by $\mathsf{EP}(f,X)$, we denote by $S_{\mathsf{EP}}(f,X)$. We can associate the dual equilibrium problem, which consists of finding a point $y^* \in X$ such that

$$f(x, y^*) \leqslant 0, \quad \forall x \in X.$$

To the solution set of the dual equilibrium problem, we denote $S^d_{\mathsf{EP}}(f,X)$. The first works in the field of equilibrium problems are attributed to Fan [14] and Nikaidô and Isoda [36]. Currently, there are many works in the area of equilibrium problems and this area, in turn, has attracted the attention of many researchers. This fact is due to the wide applicability of such problems in the most diverse areas, such as economics and mathematics. Within mathematics, we can highlight scalar and vector optimization problems, variational inequalities, saddle point problems, and Nash equilibrium problems. See [3], [34], [38], [46].

The first work to relate the proximal point method to solving an equilibrium problem was proposed Iusem and Sosa [25], in which they sought to solve an equilibrium problem for a regularized function $(\mathsf{EP}(f_k, X))$ given by

$$f_k(x, y) = f(x, y) + \gamma_k \langle x - x^k, y - x \rangle, \tag{1}$$

and $\{\gamma_k\}$ is a bounded positive auxiliary sequence. As in each iteration the solution of a regularized equilibrium problem (called a subproblem) is important, both to a theoretical and computational point of view, establish convergence results assuming which approximate solutions to the subproblems are calculated.

A problem that extends an equilibrium problem is the called quasi-equilibrium problems (QEP), Such a problem consists of finding a point $x^* \in C(x^*)$ such that

$$f(x^*, y) \geqslant 0, \quad \forall\, y \in C(x^*),$$

where $C : X \rightrightarrows X$ is a point-set map, i.e., it associates to each $x \in X$ a closed, convex, nonempty subset $C(x) \subset X$. Note that when $C(x) \equiv X$ for all $x \in X$, the quasi-equilibrium problem becomes the equilibrium problem. Quasi-equilibrium problems arise precisely to fill a gap left by equilibrium problems, since some problems cannot be modeled as equilibrium problems, but as quasi-equilibrium problems. As an example, we can cite quasi-variational inequalities and generalized Nash games, among others. We suggest reading [13], [18], [29], [46], [52],

One of the first works to propose an extension of (1) to quasi-equilibrium problems was proposed by Santos e Souza [47]. In this work, we make an extension of the work done in [47], where we propose three algorithms: In the first one, we use the Bregman distance in substitution of the Euclidean norm, and for that, we use the following regularization:

$$f_k(x^{k+1}, y) = f(x^{k+1}, y) + \gamma_k \langle \nabla\varphi(x^{k+1}) - \nabla\varphi(x^k), y - x^{k+1} \rangle$$

where $\{\gamma_k\}$ is a positive, bounded sequence and the function $\varphi : H \to \mathbb{R}$, called the Bregman function, is a differentiable function that satisfies some regularization properties . See that when $\varphi(x) = \frac{\|x\|^2}{2}$ we resume work [47].

The other two proposed algorithms are inexact versions of the proximal point method, where we initially study the convergence of the method using the regularized function:

$$f_k^e(x, y) = f(x, y) + \gamma_k \langle x - x^k, y - x \rangle - \langle e^k, y - x \rangle.$$

Again, $\{\gamma_k\}$ is a bounded positive scalar sequence. In the above regularization, $e^k \in X^*$ where X^* represents the topological dual of X. For this regularization, we perform the convergence analysis using two separate estimates for e^k:

(E1) $\|e^k\| \leqslant \|x^{k+1} - x^k\|$;

(E2) $\|x^{k+1} - x^k - e^k\| \leqslant \max\left\{\|x^{k+1} - x^k\|, \|e^k\|\right\}$.

In both cases, we obtained similar results regarding the convergence analysis.

In the second method, we will consider as the next iterate $x^{k+1} \in X$ a point close enough to the exact solution controlled by a summable sequence of error $\{\varepsilon_k\}$. While the exact solution needs to find a point belonging to $C(x^k)$ which is a solution of the equilibrium problem with the regularized bifunction f_k, this inexact version takes as its next iterate any point in an ε-neighborhood (not necessarily in $C(x^k)$) of the exact solution of the subproblem.

This work is divided as follows: In the first chapter, we bring some basic definitions and results that will appear throughout the text; in Chapters 2 and 3, we address some results on equilibrium and quasi-equilibrium problems. In Chapter 4, we study the proximal point method, in which we propose three versions: The first one, in which we make use of the Bregman Distance and the other two, where we propose inexact versions of the method to solve quasi-equilibrium problems. In Chapter 5 we apply quasi-equilibrium problems to the Cournout Model.

Chapter 1

Basic Concepts

In this chapter, we present some concepts, results, and notations that will be used throughout this text. The results presented here are widely disseminated in the related literature, and for this reason, we will not prove the results were. We will give the references in which the reader can find such proofs. Let's begginning by start by bringing some definitions and results in convex analysis, later, we will approach some results about Bregman distance, and finally Fejér and quasi-Fejér convergence.

1.1 Convex Analysis Elements

Definition 1. *A set* $X \subset \mathbb{R}^n$ *is called a convex set if for any* $a, b \in X$ *and* $\lambda \in [0, 1]$*,*

$$\lambda a + (1 - \lambda)b \in X.$$

Definition 2. *Let* $X \subset \mathbb{R}^n$ *be a convex set and* $\bar{x} \in X$*. The normal cone at the point* $\bar{x}$ *with respect to the set* X *is given by:*

$$\mathcal{N}_X(\bar{x}) = \{d \in \mathbb{R}^n | \langle d, x - \bar{x} \rangle \leqslant 0, \forall x \in X\}.$$

The (orthogonal) projection of a point $x \in \mathbb{R}^n$ onto a set $X \subset \mathbb{R}^n$ is a point of X that is closest to x. In the other words, a projection at x onto X is a global solution to the problem.

$$\min_{y \in X} \|y - x\|.$$

Next, we present the famous projection theorem.

Theorem 1. *Let $X \subset \mathbb{R}^n$ be a convex and closed set. Then for all $x \in \mathbb{R}^n$, the projection of x onto X denoted by $P_X(x)$, exists and is unique. Moreover, $\bar{x} = P_X(x)$ if and only if*

$$\bar{x} \in X, \langle x - \bar{x}, y - \bar{x} \rangle \leqslant 0 \quad \forall y \in X,$$

or equivalently, $\bar{x} \in X, x - \bar{x} \in \mathcal{N}_X(\bar{x})$.

Proof: See [26, Theorem 3.3.32]. □

Definition 3. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if its domain is a convex set and for all x, y in its domain, and all $\lambda \in [0, 1]$, we have*

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda) f(y).$$

The following theorem provides a characterization of convex differentiable functions.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a convex and open set and $f : \Omega \to \mathbb{R}$ be a differentiable function on Ω. Then the properties are equivalent:*

a) The function f is convex in Ω.

b) For every $x \in \Omega$ and every $y \in \Omega$,

$$f(y) \geqslant f(x) + \langle \nabla f(x), y - x \rangle.$$

c) For every $x \in \Omega$ and every $y \in \Omega$,

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geqslant 0.$$

If f is two differentiable over Ω, the above properties are also equivalent to

d) The Hessian matrix of f is semi-definite positive at every point of Ω:

$$\langle \text{Hess } f(x) \cdot d, d \rangle \geqslant 0 \quad \forall\ x \in \Omega, \forall\ d \in \mathbb{R}^n.$$

Proof: See [26, Theorem 3.4.30]. □

Theorem 3. *(Necessary and sufficient optimality conditions for a convex minimization problem) Let $X \subset \mathbb{R}^n$ be a convex set and $f : \Omega \to \mathbb{R}$ be a convex and differentiable function on the open set Ω containing X. Then $\bar{x}$ is a minimizer of f in X if and only if*

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geqslant 0,\ \forall\ x \in X, \tag{1.1}$$

or equivalently,

$$-\nabla f(\bar{x}) \in \mathcal{N}_X(\bar{x}).$$

Moreover, the condition (1.1) *is equivalent to*

$$\langle \nabla f(x), x - \bar{x} \rangle \geqslant 0 \ \forall \ x \in X. \tag{1.2}$$

If X *is a closed set,* (1.1) *and* (1.2) *are also equivalent to the following condition:*

$$\bar{x} = P_X(\bar{x} - \alpha \nabla f(\bar{x})) \text{ for some } \alpha > 0.$$

Proof: See [26, Theorem 3.4.37]. □

So far, results have been presented for differentiable functions. The next results are concerned with functions that are only convex.

Definition 4. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a convex function. We say that* $y \in \mathbb{R}^n$ *is a subgradient of* f *at the point* $x \in \mathbb{R}^n$ *if*

$$f(z) \geqslant f(x) + \langle y, z - x \rangle \ \forall \ z \in \mathbb{R}^n.$$

The set of all subgradients of f *at* x *is called the subdifferential of* f *in* x*; we denote it by* $\partial f(x)$.

The next result deals with the optimality condition for minimizing a convex function in a convex set and is used in works dealing with equilibrium and quasi-equilibrium problems.

Theorem 4. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a convex function and* $X \subset \mathbb{R}^n$ *be a convex set. Then* $\bar{x}$ *is a minimizer of* f *in* X *if and only if*

$$\exists \, y \in \partial f(\bar{x}); \langle y, x - \bar{x} \rangle \geqslant 0, \ \ \forall x \in X,$$

or, equivalently,

$$0 \in \partial f(\bar{x}) + \mathcal{N}_X(\bar{x}).$$

Proof: See [26, Theorem 3.4.54]. □

Other definitions that appear frequently in works about equilibrium and quasi-equilibrium are upper and lower semicontinuous and lower semicontinuous functions. Which we recall below.

Definition 5. *Let us say that* f *is upper semicontinuous at* x_0 *if for every* $\varepsilon > 0$ *there exists a neighborhood* U *of* x_0 *such that* $f(x) < f(x_0) + \varepsilon$ *for all* $x \in U$.

Definition 6. *A function is said to be lower semicontinuous at* x_0 *if for every* $\delta > 0$ *there exists a neighborhood* V *of* x_0 *such that* $f(x) > f(x_0) - \delta$ *for all* $x \in V$.

Remark 1. *Clearly, a function is continuous at* x_0 *if and only if it is both upper and lower semicontinuous there.*

1.2 Bregman distances

Let $A \subseteq \mathbb{R}^n$ be a closed and convex set with $\operatorname{int} A$ nonempty, where $\operatorname{int} A$ denotes the interior of set A. Consider a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ strictly convex, proper and lower semicontinuous with closed domain $\mathcal{D} := \operatorname{dom}(\varphi)$ and continuously differentiable on $\operatorname{int} A$.

Definition 7. *The Bregman distance associated to* φ *with zone* A *is given by*

$$D_\varphi(x, y) = \begin{cases} \varphi(x) - \varphi(y) - \langle \nabla\varphi(y), x - y\rangle, & \forall\, x \in A,\ \forall y \in \mathit{int} A \\ +\infty, \ \textit{otherwise}. \end{cases}$$

The following function are examples of Bregman distances.

Example 1. *Consider the Bregman function* $\varphi(x) = \frac{1}{2}\|x\|^2$ *and its respective Bregman distance is given by*

$$D_\varphi(x, y) = \frac{1}{2}\|x - y\|^2,$$

with $A = \mathbb{R}^n$. *Indeed,*

$$\begin{aligned} D_\varphi(x, y) &= \varphi(x) - \varphi(y) - \langle \nabla\varphi(y), x - y\rangle \\ &= \frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2 - \langle y, x - y\rangle \\ &= \frac{1}{2}\|x\|^2 - \|y\|^2 + \frac{1}{2}\|y\|^2 - \langle x - y, y\rangle \\ &= \frac{1}{2}\|x\|^2 - \langle y, y\rangle + \frac{1}{2}\|y\|^2 - \langle x - y, y\rangle \\ &= \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \langle x - y + y, y\rangle \\ &= \frac{1}{2}\|x\|^2 - \langle x, y\rangle + \frac{1}{2}\|y\|^2 \\ &= \frac{1}{2}\langle x - y, x - y\rangle \\ &= \frac{1}{2}\|x - y\|^2. \end{aligned}$$

Example 2. *The Bregman function* $\varphi(x) = -\sum_{i\in I(x)}^{n} \log x_i$ *and its respective Bregman distance*

$$D_\varphi(x,y) = \sum_{i\in I(x)}^{n} \left(\log(y_i/x_i) + \frac{x_i}{y_i} - 1\right)$$

with $A = \mathbb{R}^n_+$, *and* $I(x) = \{i : x_i > 0\}$. *(This function is called Burg entropy).*

Indeed,

$$\begin{aligned}
D_\varphi(x,y) &= \varphi(x) - \varphi(y) - \langle \nabla\varphi(y), x - y\rangle \\
&= -\sum_{i\in I(x)}^{n} \log(x_i) - \left(-\sum_{i\in I(x)}^{n} \log(y_i)\right) - \langle -\frac{1}{y_i}, x_i - y_i\rangle \\
&= -\sum_{i\in I(x)}^{n} \log(x_i) - \left(-\sum_{i\in I(x)}^{n} \log(y_i)\right) - \left(-\frac{x_i}{y_i} + 1\right) \\
&= \sum_{i\in I(x)}^{n} \log(y_i) - \sum_{i\in I(x)}^{n} \log(x_i) + \frac{x_i}{y_i} - 1 \\
&= \sum_{i\in I(x)}^{n} \left(\log(y_i/x_i) + \frac{x_i}{y_i} - 1\right).
\end{aligned}$$

Example 3. *The Bregman function* $\varphi(x) = \sum_{i=1}^{n} x_i \log x_i$ *called Shannon entropy and its respective Bregman distance*

$$D_\varphi(x,y) = \sum_{i=1}^{n} [x_i \log(x_i/y_i) + y_i - x_i]$$

with $A = \mathbb{R}^n_+$ *known as Kullback-Leibler distance.*

Indeed,

$$\begin{aligned}
D_\varphi(x,y) &= \varphi(x) - \varphi(y) - \langle \nabla\varphi(y), x - y\rangle \\
&= \sum_{i=1}^{n} x_i \log(x_i) - \sum_{i=1}^{n} y_i \log(y_i) + \sum_{i=1}^{n} (\log(y_i) + 1)(y_i - x_i) \\
&= \sum_{i=1}^{n} x_i \log(x_i) - \sum_{i=1}^{n} y_i \log(y_i) + \sum_{i=1}^{n} \log(y_i)(y_i - x_i) + y_i - x_i \\
&= \sum_{i=1}^{n} x_i \log(x_i) - \sum_{i=1}^{n} x_i \log(y_i) + \sum_{i=1}^{n} y_i \log(y_i) - \sum_{i=1}^{n} y_i \log(y_i) + y_i - x_i \\
&= \sum_{i=1}^{n} x_i \log(x_i/y_i) + y_i - x_i.
\end{aligned}$$

Next, we state the well-known three-point property for Bregman distances. More information on Bregman functions and distances can be found, for example, in the recent paper by Reem et al. [43].

For any $x \in \mathcal{D}$ and $y, z \in \operatorname{int}\mathcal{D}$, it is straightforward to check that

$$\langle \nabla\varphi(y) - \nabla\varphi(z), z - x\rangle = D_\varphi(x, y) - D_\varphi(x, z) - D_\varphi(z, y). \tag{1.3}$$

Following Burachik and Scheimberg [6], we consider throughout this work the following set of assumptions on φ:

Assumptions 2.

i) The right level sets of $D_\varphi(y, \cdot)$*:*

$$S_{y,\alpha} := \{z \in int\mathcal{D} \ : \ D_\varphi(y, z) \leqslant \alpha\}$$

are bounded for all $\alpha \geqslant 0$ *and for all* $y \in \mathcal{D}$*.*

ii) If $\{x^k\}, \{y^k\} \subset int\mathcal{D}$ *with* $\lim_{k\to+\infty} x^k = x$*,* $\lim_{k\to+\infty} y^k = x$ *and*

$$\lim_{k\to+\infty} D_\varphi(x^k, y^k) = 0,$$

then

$$\lim_{k\to+\infty} D_\varphi(x, x^k) - D_\varphi(x, y^k) = 0.$$

iii) If $\{x^k\} \subset \mathcal{D}$ *is bounded,* $\{y^k\} \subset int\mathcal{D}$ *is such that* $\lim_{k\to+\infty} y^k = y$ *and* $\lim_{k\to+\infty} D_\varphi(x^k, y^k) = 0$*, then* $\lim_{k\to+\infty} x^k = y$*.*

iv) For every $y \in A$*, there exists* $x \in int\mathcal{D}$ *such that* $\nabla\varphi(x) = y$*.*

Remark 2. *The Bregman distances in Examples 1-3 are examples of functions that satisfy Assumption 2; see [5].*

1.3 Fejér and quasi-Fejér Convergence

Now, we recall some important results concerning sequences in equilibrium and quasi-equilibrium problems.

Definition 8. *A sequence* $\{z^k\}$ *is quasi-Fejér convergent to a set* U *if, for each* $u \in U$*, there exists a non-negative sequence* $\{\epsilon_k\}$ *with* $\sum_{k=0}^{+\infty} \epsilon_k < +\infty$ *such that*

$$\|z^{k+1} - u\|^2 \leqslant \|z^k - u\|^2 + \epsilon_k, \quad \forall k \in \mathbb{N}.$$

We say that $\{z^k\}$ *is Fejér convergent to a nonempty set* U *if, for all* $k \in \mathbb{N}$*,*

$$\|z^{k+1} - x\| \leqslant \|x^k - x\|, \quad \forall x \in U. \tag{1.4}$$

An important property of sequences that are Fejer and quasi-Fejér convergent is that they are bounded. Furthermore, in some cases, it is possible to show that all sequences converge. This fact is evidenced in the next result.

Lemma 1. *Let U be a nonempty set and assume that* $\{z^k\}$ *is quasi-Fejér convergent to U. Then,* $\{z^k\}$ *is bounded. Moreover, if a cluster point* z *of* $\{z^k\}$ *belongs to U, then* $\{z^k\}$ *converges weakly to* z*.*

Proof: See [50, Proposition 2.1]. □

Lemma 2. *Let* U *be a nonempty set and assume that* $\{z^k\}$ *is Fejer convergent to U. So* $\{z^k\}$ *is bounded. Furthermore, if* z *is a weak accumulation point of* $\{z^k\}$ *that belongs to U, then* $\{z^k\}$ *weakly converges to* z*.*

Proof: See [1, Proposition 1]. □

Lemma 3. *Let* $\{\nu_k\}, \{\gamma_k\}$ *and* $\{\beta_k\}$ *be nonnegative sequences of real numbers satisfying* $\nu_{k+1} \leqslant (1+\gamma_k)\nu_k + \beta_k$ *and such that* $\sum_{k=0}^{+\infty} \gamma_k < +\infty, \sum_{k=0}^{+\infty} \beta_k < +\infty$*. Then, the sequence* $\{\nu_k\}$ *converges.*

Proof: See [40, Lemma 2.2.2]. □

Chapter 2

Equilibrium Problems

Throughout this chapter, $X \subset H$ will denote a convex, closed, nonempty set and H a Hilbert space. We will approach the main concepts and results of Equilibrium Problems. The equilibrium problem (EP) consists of finding $x^* \in X$ such that

$$f(x^*, y) \geqslant 0, \quad \forall y \in X. \tag{2.1}$$

This problem will be denoted by $\mathsf{EP}(f, X)$ and its solution set by $S_{\mathsf{EP}}(f, X)$. A problem related to $\mathsf{EP}(f, X)$ is finding $y^* \in X$ such that

$$f(x, y^*) \leqslant 0, \quad \forall x \in X. \tag{2.2}$$

This problem is called the dual of $\mathsf{EP}(f, X)$ and its solution set is denoted by $S^d_{\mathsf{EP}}(f, X)$. One of the first works to propose a version of the proximal point method to solve equilibrium problems was Iusem and Sosa in [25]. In [25], an algorithm is proposed, which is denoted by PPEP and at each iteration solves an equilibrium problem and is described below:
Take a sequence of regularization parameters: $\{\gamma_k\} \subset (\theta, \bar{\gamma}]$, for some $\bar{\gamma} > \theta$. Choose $x^0 \in X$ and construct the sequence $\{x^k\} \subset X$ as follows:
Given x^k, we choose x^{k+1} as the unique solution of the problem $\mathsf{EP}(f_k, X)$, where $f_k : X \times X \to \mathbb{R}$ is defined as

$$f_k(x, y) = f(x, y) + \gamma_k \langle x - x^k, y - x \rangle.$$

The results proves the well definition of the proposed method, and as the convergence results will be presented below. Throughout this chapter, H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\|x\| = \sqrt{\langle x, x \rangle}$, for all $x \in H$. Let $X \subset H$ be a closed and convex set and a bifunction $f : X \times X \to \mathbb{R}$ satisfying the following proprieties:

Assumptions 3.

i) $f(x, x) = 0$ *for all* $x \in X$.

ii) $f(\cdot, y) : X \to \mathbb{R}$ *is upper semicontinuous for all* $y \in X$.

iii) $f(x, \cdot)$ *is convex and lower semicontinuous for all* $x \in X$.

iv) f *is monotone if* $f(x, y) + f(y, x) \leqslant 0$ *for all* $x, y \in X$.

v) For any sequence $\{z^n\}_{n\in\mathbb{N}} \subset X$ *with* $\lim\limits_{n\to+\infty} \|z^n\| = +\infty$*, there exists* $u \in X$ *and* $n_0 \in \mathbb{N}$ *such that* $f(z^n, u) \leqslant 0$ *, for all* $n \geqslant n_0$.

The first result in the chapter deals with the relationship between equilibrium problems and the dual equilibrium problem.

Proposition 1. *Under conditions* i)-iv) *it holds that* $S_{EP}(f, X) = S^d_{EP}(f, X)$.

Proof: See [23, proposition 3.3]. □

The next result says that when we add hypothesis v) in Assumption 3, it is guaranteed that the equilibrium problem has a solution.

Proposition 2. *Assume that* f *satisfies the Assumptions 3. Then* $S_{EP}(f, X)$ *is nonempty.*

Proof: See [23, Proposition 4.2]. □

One of the most simple and popular strategies for solving EP is the so-called regularization method. The Tikhonov regularization method for ill-posed problems is well-known for minimization, monotone inclusion, and fixed-point problems. This approach was introduced for solving equilibrium problems by [31]; see also [33] and [32]. This method solves at each iteration the regularized $EP(f_k, X)$, where f_k is given by

$$f_k(x, y) = f(x, y) + \gamma_k \langle x - x^k, y - x\rangle \tag{2.3}$$

The next result guarantees that under certain hypotheses, the regularized problem has a solution, which in turn is unique.

Proposition 3. *Take* f *satisfying Assumptions 3. Then* $EP(f_k, X)$ *where* f_k *is given for* (2.3) *has a unique solution.*

Proof: See [25, Proposition 3]. □

Remark 3. *The previous result is used in order to guarantee that the method proposed is well defined, since, in each iteration,* x^{k+1} *is the only solution of the regularized equilibrium problem.*

The next result establishes a relationship between the elements belonging to the solution set of the dual problem and the solution set of the regularized problem. Consider a variation of the function (2.3) which we will denote by

$$\tilde{f}(x, y) = f(x, y) + \gamma \langle x - \bar{x}, y - x \rangle,$$

where $\gamma > 0$ and $\bar{x} \in X$.

Proposition 4. *Assume that f satisfies Assumptions 3* i), ii) *and* iii). *If* $\tilde{x} \in S(\tilde{f}, X)$ *and* $x^* \in S^d(f, X)$ *then*

$$\|\tilde{x} - x^*\|^2 + \|\bar{x} - \tilde{x}\|^2 \leqslant \|\bar{x} - x^*\|^2.$$

Proof: See [25, Proposition 4]. □

Next, we present the proximal point method proposed by Iusem and Sosa in [25], as well as the main convergence result. To this end, we assume that f is monotone.

Algorithm 1 Proximal Point Equilibrium Problem

1: Take a positive and bounded sequence of regularization parameters $\{\gamma_k\}$.
2: Choose a $x^0 \in X$ and construct the sequence $\{x^k\}$ as follows:
3: Given x^k, x^{k+1} is the unique solution of the problem $EP(f_k, X)$, where $f_k(x, y)$ is given by (2.3).

Theorem 5. *Consider* $EP(f, X)$, *where* f *satisfies Assumption 3:* i), ii) *and* iii). *For all* $x^0 \in X$,

*a) If f satisfies Assumption 3,*iv). *then the sequence* $\{x^k\}$ *generated by Algorithm 1 is well-defined;*

b) if $S^d(f, X) \neq \emptyset$; *then the sequence* $\{x^k\}$ *is bounded and* $\lim_{k \to +\infty} \|x^{k+1} - x^k\| = 0$.

c) under the assumptions of items i) *and* ii) *the sequence* $\{x^k\}$ *is an asymptotically solving sequence for* $EP(f, X)$,*i.e.,* $\lim_{k \to +\infty} f(x^k, y) \geqslant 0$, *for all* $y \in X$.

d) *if additionally* $f(\cdot, y)$ *is weakly upper semicontinuous for all* $y \in X$*, then all weak cluster points of* $\{x^k\}$ *solve* $EP(f, X)$*,*

e) *if additionally* $S(f, X) = S^d(f, X)$ *then the sequence* $\{x^k\}$ *is weakly convergent to some solution* $\widehat{x}$ *of* $EP(f, X)$*.*

Proof: See [25, Theorem 1]. □

Other essential work that was studied by us and that presents an version of the proximal point method was proposed by Muh and Quoc in [35] in which the proposed algorithm is used in numerical experiments. Next we present the main assumption used, as well as their main convergence result.

Definition 9. *Let* $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ *is said to be strongly monotone on* X *with modulus* $\tau > 0$ *if* $f(x, y) + f(y, x) \leqslant -\tau\|x - y\|^2$*, for all* $x, y \in X$*.*

Lipschitz-type: There exists constants L_1 and L_2 such that

$$f(x, y) + f(y, z) \geqslant f(x, z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \quad x, y, z \in X. \tag{2.4}$$

Assume that f is strongly monotone on X with modulus $\tau > 0$ and satisfies (2.4). Choose a tolerance $\varepsilon \geqslant 0$ and $0 < \rho \leqslant 1/(L_2)$.

Algorithm 2 : Strongly Monotone Problem

1: Choose $x^0 \in X$ and construct the sequence $\{x^k\}$ as follows:

2: If $\|x^{k+1} - x^k\| \leqslant \varepsilon(1 - r)/r$, with $r := \sqrt{1 - 2\rho(\tau - L_1)}$, then termine: x^{k+1} is an ε-solution to (EP). Otherwise, increase k by 1 and go to iteration k.

Theorem 6. *Suppose that* f *is strongly monotone on X with modulus* $\tau > 0$ *and satisfies the Lipschitz-type condition* (2.4)*. Then, for any starting point* x^0*, the sequence* $\{x^k\}$ *defined by*

$$x^{k+1} = \arg\min_{y \in X}\{\rho f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\}$$

satisfies

$$\|x^{k+1} - x^*\|^2 \leqslant \alpha\|x^k - x^*\|^2, \quad k \geqslant 0,$$

provided $0 < \rho < 1/(2L_2)$, where x^* is the unique solution of (EP) and $\alpha := 1 - 2\rho(\tau - L_1)$.

Proof: See [35, Theorem 2.1]. □

Still in the area of equilibrium problems, some authors have developed inexact methods of the proximal point method to solve equiibrium problems. We highlight the work done by Iusem and Nasri [24], which was the basis for the production of this work and we describe it below

Definition 10. *The modulus of total convexity of g at* x *is the function* $\nu_\varphi : H \times [0, +\infty) \to [0, +\infty]$ *defined by* $\nu_\varphi(x, t) = \inf\{D_\varphi(y, x) : \|y - x\| = t\}$.

Next we present some assumptions on g that will be needed in our convergence analysis.

(H1) The level sets of $D_\varphi(x, \cdot)$ are bounded for all $x \in H$.

(H2) $\inf_{x \in M} \nu_\varphi(x, t) > 0$, for all bounded set $M \subset H$ and $t > 0$.

(H3) $\nabla\varphi$ is uniformly continuous on bounded subsets of H.

(H4) $\nabla\varphi$ is onto, i.e., for all $y \in H^*$, exists $x \in H$ such that $\nabla\varphi(x) = y$.

(H5) $\lim_{\|x\| \to +\infty} \left[\varphi(x) - \rho\|x - z\|\right] = +\infty$ for all $z \in X$ fixed and $\rho \geqslant 0$.

(H6) If $\{y^j\}$ e $\{z^j\}$ are sequences in X that converge weakly to y and z, respectively, and $y \neq z$, then
$$\liminf_{j \to +\infty} |\langle \nabla\varphi(y^j) - \nabla\varphi(z^j), y - z\rangle| > 0.$$

These properties, with the exception of H5, were identified in [21].

Next, we present the two inexact algorithms proposed by Iusem and Nasri in [24] and then their main convergence results.

For the next theorem we will consider a set of hypotheses adopted in that work, which we will call the set of Assumptions 4

Assumptions 4.

i) $f(x, x) = 0$ *for all* $x \in X$.

ii) $f(x, \cdot) : X \to \mathbb{R}$ *is convex and lower semicontinuous for all* $x \in X$.

iii) $f(\cdot, y) : X \to \mathbb{R}$ *is upper semicontinuous for all* $y \in X$,

iv) Exists $\theta \geqslant 0$ *such that* $f(x, y) + f(y, x) \leqslant \theta\langle \nabla\varphi(x) - \nabla\varphi(y), x - y\rangle$ *for all* $x, y \in X$.

Algorithm 3 :Inexact Proximal Point + Bregman Projection Method

1: Choose a $x^0 \in X$;

2: Given x^j, find a pair $\tilde{x}^j \in H, e^j \in H^*$ such that $\tilde{x}^j$ solves $EP(f_j^e, X)$ with

$$f_j^e(x,y) = f(x,y) + \gamma_j \langle \nabla\varphi(x) - \nabla\varphi(x^j), y - x\rangle - \langle e^j, y - x\rangle \tag{2.5}$$

i.e.,

$$f_j^e(\tilde{x}^j, y) \geqslant 0 \ \forall y \in X,$$

and e^j satisfies

$$\|e^j\|_* \leqslant \begin{cases} \sigma\gamma_j D_\varphi(\tilde{x}^j, x^j), & \mathrm{if}\|\tilde{x}^j - x^j\| < 1 \\ \sigma\gamma_j \nu_\varphi(x^j, 1), & \mathrm{if}\|\tilde{x}^j - x^j\| \geqslant 1, \end{cases}$$

with D_φ and ν_φ as in the definition given below

3: Let

$$\upsilon^j = \gamma_j[\nabla\varphi(x^j) - \nabla\varphi(\tilde{x}^j)] + e^j. \tag{2.6}$$

If $\upsilon^j = 0$ or $\tilde{x}^j = x^j$, then stop. Otherwise, take $H_j = \{x \in H : \langle \upsilon^j, x - \tilde{x}^j\rangle = 0\}$ and define

$$x^{j+1} = \arg\min_{x\in H_j} D_\varphi(x, x^j).$$

Algorithm 4 : :Inexact Proximal Point– Extragradient Method

1: Choose $x^0 \in X$

2: Given x^j, find a pair $\tilde{x}^j \in H, e^j \in H^*$ such that $\tilde{x}^j$ solves $EP(f_j^e, X)$ with

$$f_j^e(x,y) = f(x,y) + \gamma_j \langle \nabla\varphi(x) - \nabla\varphi(x^j), y - x\rangle - \langle e^j, y - x\rangle \tag{2.7}$$

i.e.,

$$f_j^e(\tilde{x}^j, y) \geqslant 0 \ \forall y \in X,$$

and e^j satisfies

$$D_\varphi(\tilde{x}^j, \nabla\varphi^{-1}[\nabla\varphi(\tilde{x}^j) - \gamma_j^{-1}e^j]) \leqslant \sigma D_\varphi(\tilde{x}^j, x^j)$$

3: If $\tilde{x}^j = x^j$, then stop. Otherwise,

$$x^{j+1} = \nabla\varphi^{-1}[\nabla\varphi(\tilde{x}^j) - \gamma_j^{-1}e^j]. \tag{2.8}$$

Theorem 7. *Consider Algorithm 3, Assumptions 4 and* $\mathsf{EP}(f, X)$*. Take* $\varphi : H \to \mathbb{R}$ *satisfying* H1−H5 *and an exogenus sequence* $\{\gamma_j\} \subset (\theta, \bar{\gamma}]$*, where* θ *is the undermonotonicity constant. Let* $\{x^j\}$ *be the sequence generated by Algorithm 3. If* $\mathsf{EP}(f, X)$ *has solutions, then*

i) $\{\tilde{x}^j\}$ *is an asymptotically solving sequence for* $\mathsf{EP}(f, X)$*.*

ii) If $f(\cdot, y)$ *is weakly upper semicontinuous for all* $y \in X$*, then all cluster points of* $\{x^j\}$ *solve* $\mathsf{EP}(f, X)$*.*

iii) If in addition either φ *satisfies H6 or* $\mathsf{EP}(f, X)$ *has a unique solution, then the whole sequence* $\{x^j\}$ *is weakly convergent to some solution* x^* *of* $\mathsf{EP}(f, X)$*.*

Proof: See [24, Theorem 5.5]. □

Theorem 8. *Consider Algorithm 4, Assumptions 4 and* $\mathsf{EP}(f, X)$*. Take* $\varphi : H \to \mathbb{R}$ *satisfying* H1−H5 *and an exogenus sequence* $\{\gamma_j\} \subset (\theta, \bar{\gamma}]$*, where* θ *is the undermonotonicity constant. Let* $\{x^j\}$ *be the sequence generated by Algorithm 4. If* $\mathsf{EP}(f, X)$ *has solutions, then*

i) $\{\tilde{x}^j\}$ *is an asymptotically solving sequence for* $\mathsf{EP}(f, X)$*.*

ii) If $f(\cdot, y)$ *is weakly upper semicontinuous for all* $y \in X$*, then all cluster points of* $\{x^j\}$ *solve* $\mathsf{EP}(f, X)$*.*

iii) If in addition either φ *satisfies* H6 *or* $\mathsf{EP}(f, X)$ *has a unique solution, then the whole sequence* $\{x^j\}$ *is weakly convergent to some solution* x^* *of* $\mathsf{EP}(f, X)$*.*

Proof: See [24, Theorem 5.8]. □

Chapter 3

Quasi-Equilibrium Problems

The quasi-equilibrium problem (shortly, QEP) consists of finding $x^* \in C(x^*)$ such that

$$f(x^*, y) \geqslant 0, \quad \forall y \in C(x^*),$$

where $C : X \rightrightarrows X$ is a point-set map, i.e., it associates to each $x \in X$ a closed, convex, non-empty subset $C(x) \subset X$. These assumptions have been used in algorithms for QEP's; see [2, 49, 51]. Furthermore, we suppose the continuity of the multivalued mapping C in the sense of Mosco (M-continuous). Related bellow

Definition 11. *Let us recall that C is said to be M-continuous if:*

(i) For $\{x^k\}, \{y^k\} \subset X$ with $y^k \in C(x^k)$, $x^k \rightharpoonup x$ and $y^k \rightharpoonup y$ implies that $y \in C(x)$, which means that the graph of C is sequentially closed.

(ii) For any sequence $\{x^k\} \subset X$ with $x^k \rightharpoonup x$ and for each $y \in C(x)$ there exists a sequence $\{y^k\} \subset X$ with $y^k \in C(x^k)$ such that $y^k \to y$.

We denote the solution set of QEP as $S_{QEP}(f, C)$. Next, we consider a set $S^* \subset S_{QEP}(f, C)$ and we assume that $S^* \neq \emptyset$, where it is given by

$$S^* = \left\{ x \in \bigcap_{z \in X} C(z) \ : \ f(x, y) \geqslant 0, \quad \forall y \in \bigcup_{z \in X} C(z) \right\}.$$

This assumption was considered to study the convergence of extragradient algorithms for solving QEP (see Strodiot et al.[49]), a projection-like method for QVIP (see [53]) and generalized Nash equilibrium problem (see [19]).

Remark 4. *In equilibrium problems, the hypothesis "the solution set of the Equilibrium Problem is nonempty" has been assumed as a standard assumption. In Quasi Equilibrium Problem, the assumption* $S^* \neq \emptyset$ *can be viewed as a natural extension to QEP's of the assumption* $S_{EP}(f, X) \neq \emptyset$ *because if* $C(x) \equiv X$*, for all* $x \in X$*, then* $\bigcap_{z \in X} C(z) = \bigcup_{z \in X} C(z) = X$ *and* $S^* = S_{EP}(f, X)$*. It is easy to verify that the assumption* $S^* \neq \emptyset$ *guarantees that* $S_{QEP}(f, C) \neq \emptyset$*. For QVIP, there are available a great number of results when either* X *is bounded or the operator* C *satisfies certain coercivity condition. However, as remarked by Giannessi and Khan [17] many applications deal with QVIP with non-coercive operators defined on unbounded sets.*

Throughout this chapter, the function $f : X \times X \to \mathbb{R}$ satisfies the Assumptions 1. The next result is a QEP version of a proposition quite useful in equilibrium problems.

Proposition 5. *Let* $\bar{x} \in X$ *be an arbitrary point and* $\widehat{x}, x^* \in X$ *such that* $\widehat{x} \in S_{EP}(\widehat{f}, C(\bar{x}))$ *and* $x^* \in S^d(f, C(\bar{x}))$ *, where* $\widehat{f}$ *is given by* $\widehat{f}(x, y) = f(x, y) + \gamma\langle x - \bar{x}, y - x\rangle$*, for some* $\gamma > 0$*. If* f *satisfies Assumptions 1, then*

$$\|\widehat{x} - x^*\|^2 + \|\bar{x} - \widehat{x}\|^2 \leqslant \|\bar{x} - x^*\|^2.$$

Proof: Take $\widehat{x} \in S_{EP}(\widehat{f}, C(\bar{x}))$ and $x^* \in S^d(f, C(\bar{x}))$. Since that $\widehat{x} \in S_{EP}(\widehat{f}, C(\bar{x}))$ we have

$$0 \leqslant \widehat{f}(\widehat{x}, y) = f(\widehat{x}, y) + \gamma\langle \widehat{x} - \bar{x}, y - \widehat{x}\rangle, \quad \forall y \in C(\bar{x}),$$

and therefore

$$-f(\widehat{x}, y) \leqslant \gamma\langle \widehat{x} - \bar{x}, y - \widehat{x}\rangle, \quad \forall y \in C(\bar{x}). \tag{3.1}$$

Now, since that $x^* \in S^d(f, C(\bar{x}))$ we have that $f(y, x^*) \leqslant 0$ for all $y \in C(\bar{x})$. In particular, for $y = \widehat{x}$ we have

$$f(\widehat{x}, x^*) \leqslant 0. \tag{3.2}$$

Make $y = x^*$ in (3.1), we have that

$$0 \leqslant \gamma\langle \widehat{x} - \bar{x}, x^* - \widehat{x}\rangle. \tag{3.3}$$

From equality of three points, we have

$$\gamma\langle \widehat{x} - \bar{x}, x^* - \widehat{x}\rangle = \frac{\gamma}{2}\left(\|x^* - \bar{x}\|^2 - \|\widehat{x} - \bar{x}\|^2 - \|x^* - \widehat{x}\|^2\right). \tag{3.4}$$

Replace (3.4) in (3.3) and using that $\gamma > 0$, we have that

$$\|x^* - \widehat{x}\|^2 + \|\widehat{x} - \bar{x}\|^2 \leqslant \|x^* - \bar{x}\|^2,$$

which proves the proposition. $\square$

Let $f : X \times X \to \mathbb{R}$ be a bifunction. We denote by $\partial_2 f$ the subdifferential (in the sense of convex analysis) of f with respect to its second argument evaluated at a point $(x, x) \in X \times X$. In view of assumptions 1, i) and iii), one has

$$\begin{aligned} \partial_2 f(x,x) &= \{v \in H : f(x,y) \geqslant f(x,x) + \langle v, y-x \rangle, \quad \forall y \in X\} \\ &= \{v \in H : f(x,y) \geqslant \langle v, y-x \rangle, \quad \forall y \in X\}. \end{aligned}$$

It is called subdifferential diagonal; see [22]. Given $x \in X$ we denote by

$$\Upsilon(x) = \|x - \arg\min_{y \in C(x)} \{f(x,y) + \frac{1}{2}\|y-x\|^2\}\|.$$

The next proposition will be used in numerical experiments. It measures the quality of a candidate's solution to QEP. It is an extension to QEP of a well known characterization of a solution of an equilibrium problem; see [22].

Proposition 6. *A point* $x^* \in S_{QEP}(f, C)$ *if only if* $\Upsilon(x^*) = 0$.

Proof: If $x^* \in S_{QEP}(f, C)$, then $x^* \in C(x^*)$ and $f(x^*, y) \geqslant 0$ for all $y \in C(x^*)$. So, we have

$$f(x^*, y) + \frac{1}{2}\|y - x^*\|^2 \geqslant f(x^*, y) \geqslant 0 = f(x^*, x^*) + \frac{1}{2}\|x^* - x^*\|^2, \quad \forall y \in C(x^*).$$

Thus, $x^* = \arg\min_{y \in C(x)} \{f(x,y) + \frac{1}{2}\|y-x\|^2$ and $\Upsilon(x^*) = 0$. Now, suppose that $\Upsilon(x^*) = 0$, that is, $x^* = \arg\min_{y \in C(x)} \{f(x,y) + \frac{1}{2}\|y-x\|^2$. From the first-order optimality condition, we have that $x^* \in C(x^*)$ and

$$0 \in \partial_2 f(x^*, x^*) + \mathcal{N}_{C(x^*)}(x^*),$$

where $\mathcal{N}_{C(x^*)}(x^*)$ stands to the normal cone to $C(x^*)$ at x^*. Thus, there exists $s^* \in \partial_2 f(x^*, x^*)$ such that $0 \in s^* + \mathcal{N}_{C(x^*)}(x^*)$, therefore $-s^* \in \mathcal{N}_{C(x^*)}(x^*)$. Follows from the definition of a normal cone that

$$\langle y - x^*, -s^* \rangle \leqslant 0, \quad \forall y \in C(x^*)$$

consequently

$$\langle y - x^*, s^* \rangle \geqslant 0, \quad \forall y \in C(x^*). \tag{3.5}$$

On the other hand, $s^* \in \mathcal{N}_{C(x^*)}(x^*)$ and from iii) $f(x^*, \cdot)$ is convex, then

$$f(x^*, y) \geqslant f(x^*, x^*) + \langle y - x^*, s^* \rangle = \langle y - x^*, s^* \rangle \geqslant 0, \quad \forall y \in C(x^*),$$

where the equality follows from i) and the second inequality is due to (3.5). $\square$

3.1 Proximal point method (PPM)

In this section, we present the version of the proximal point method proposed by [47] for solving a QEP.

Algorithm 5 : Proximal Point Method

1: Take a bounded auxiliary sequence of positive parameters $\{\gamma_k\}$ and choose $x^0 \in X$.

2: Given x^k, compute

$$x^{k+1} \in S_{EP}(f_k, C_k),$$

where $f_k(x,y) = f(x,y) + \gamma_k \langle x - x^k, y - x\rangle$ and $C_k = C(x^k)$.

3: Se $x^{k+1} = x^k$, stop. Otherwise, set $k = k+1$ and return to previous step.

It is important to ensure that the method is well defined i.e., given x^k generates x^{k+1}. We will show that the algorithm proposed above is indeed feasible and well defined. We will show that assuming that the function f satisfies the conditions $i) - v)$ in assumptions 1, the function f_k also satisfies the conditions.

Proposition 7. *If* f *satisfies Assumptions 1:* $i)$-$iv)$*, then* f_k *satisfies* $i)$-$v)$.

Proof: First, let's see that if f satisfies the properties $i)$-$iii)$, the same happens with the regularized function $f_k(x,y) = f(x,y) + \gamma_k\langle x - x^k, y - x\rangle$. Clearly, $f_k(x,x) = 0$ for all $x \in X$, which tells us that $i)$ is satisfied. Since f satisfies $ii)$, $\gamma_k\langle x - x^k, y - x\rangle$ is a multilinear map, given sequences $\{x^j\}$, $\{y^j\} \subset X$ such that $x^j \rightharpoonup x$ and $y^j \rightharpoonup y$, then $f_k(x^j, y^j) \to f_k(x,y)$. Finally, let's see that the function $f_k(x,\cdot)$ is convex. Let $a, b \in X$ and $t \in [0,1]$. Thus

$$\begin{aligned}
f_k(x, ta + (1-t)b) &= f(x, ta + (1-t)b) + \gamma_k\langle x - x^k, ta + (1-t)b - x\rangle \\
&= f(x, ta + (1-t)b) + \gamma_k\langle x - x^k, ta - tx + tx + (1-t)b - x\rangle \\
&\leqslant tf(x,a) + (1-t)f(x,b) + t\gamma_k\langle x - x^k, a - x\rangle + (1-t)\gamma_k\langle x - x^k, b - x\rangle \\
&= t\left(f(x,a) + \gamma_k\langle x - x^k, a - x\rangle\right) + (1-t)\left(f(x,b) + \gamma_k\langle x - x^k, b - x\rangle\right) \\
&= tf_k(x,a) + (1-t)f_k(x,b).
\end{aligned}$$

In the inequality, it was stated that the function $f(x,\cdot)$ is convex. See also that if f is monotone, that is, $f(x,y)+f(y,x)\leqslant 0$, then the same goes for f_k.

$$\begin{aligned}
f_k(x,y)+f_k(y,x) &= f(x,y)+\gamma_k\langle x-x^k, y-x\rangle + f(y,x)+\gamma_k\langle y-x^k, x-y\rangle. \\
&= f(x,y)+f(y,x)+\gamma_k\langle x-x^k, y-x\rangle+\gamma_k\langle x^k-y, y-x\rangle. \\
&\leqslant \gamma_k\langle x-y, y-x\rangle \\
&= -\gamma_k\|x-y\|^2 \\
&\leqslant 0.
\end{aligned}$$

In the first inequality, the fact that $f(y,x)+f(y,x)\leqslant 0$ was used. In summary, we have that f_k satisfies property iv), whenever f also satisfies. Finally, we will show below that f_k also satisfies the v) property. To prove the validity of the property v) for the function f_k, let's take a sequence $\{z^k\}$ such that $\lim_{k\to\infty}\|z^k\|=+\infty$, and $\gamma_k>0$. We claim that v) is valid when $u=P_X(x^k)$, where $P: H\to X$ denotes the orthogonal projection onto X. Note that

$$\begin{aligned}
f_k(z^k,u) &= f(z^k,P_X(x^k))-\gamma_k\langle z^k-x^k, z^k-P_X(x^k)\rangle \\
&= f(z^k,P_X(x^k))-\gamma_k\langle z^k-P_X(x^k), z^k-P_X(x^k)\rangle-\gamma_k\langle P_X(x^k)-x^k, z^k-P_X(x^k)\rangle \\
&\leqslant -f(P_X(x^k),z^k)+\gamma_k\langle x^k-P_X(x^k), z^k-P_X(x^k)\rangle-\gamma_k\langle z^k-P_X(x^k), z^k-P_X(x^k)\rangle \\
&\leqslant -f(P_X(x^k),z^k)-\gamma_k\|z^k-P_X(x^k)\|^2, \qquad (3.6)
\end{aligned}$$

using the definition of f_k in the first equality, the fact that $\{z^k\}\subset X$ together with the well-known obtuse angle property of orthogonal projections, in the first inequality and iv) in the second inequality. We introduce now some notation for the marginals of f. For each $x\in X$, define $g_x: X\to\mathbb{R}$ as

$$g_x(y)=f(x,y) \qquad (3.7)$$

Take $\widehat{x}\in ri(X)$, so that $\widehat{x}$ belongs to the relative interior of the effective domain of g_u. Since g_u is convex by iii), its subdifferential at $\widehat{x}$, namely $\partial_2 g_u(\widehat{x})$, is nonempty. Take $\widehat{v}\in\partial_2 g_u(\widehat{x})$. By the definition of subdifferential,

$$\langle\widehat{v}, z^k-\widehat{x}\rangle\leqslant g_u(z^k)-g_u(\widehat{x})=f(u,z^k)-f(u,\widehat{x}) \qquad (3.8)$$

In view of (3.8)

$$
\begin{aligned}
-f_k(u, z^k) &\leqslant \langle \widehat{v}, z^k - \widehat{x} \rangle - f(u, \widehat{x}) \\
&\leqslant \|\widehat{v}\| \|z^k - \widehat{x}\| - f(u, \widehat{x}) \\
&\leqslant \|\widehat{v}\| \|z^k - u\| + \|\widehat{v}\| \|u - \widehat{x}\| - f(u, \widehat{x}).
\end{aligned} \tag{3.9}
$$

Replacing (3.9) in (3.6),

$$
f_k(z^k, u) \leqslant \|z^k - u\| \left[\|\widehat{v}\| - \gamma_k \|z^k - u\| \right] + \|\widehat{v}\| \|u - \widehat{x}\| - f(u, \widehat{x}). \tag{3.10}
$$

Since that $\gamma_k > 0$ and $\lim_{k \to +\infty} \|z^k\| = +\infty$, so that $\lim_{k \to +\infty} \|z^k - u\| = +\infty$, it follows easily from (3.10) that $\lim_{k \to +\infty} f_k(z^k, u) = -\infty$, so that $f_k(z^k, u) \leqslant 0$ for large enough k. We have verified that f_k satisfies the assumptions $i)$-$v)$ and proved that it is finished. $\square$

Remark 5. *Since that f_k satisfies i)-v) for each $k \in \mathbb{N}$, x^{k+1} is well defined, i.e. $S_{EP}(f_k, C_k)$ is nonempty due to Proposition 2. Therefore, the well-defined sequence follows from the existence result for EP given by Proposition 2 taking into account that f_k satisfies $i)$-$v)$ and $C_k = C(x^k)$ is a nonempty, closed, and convex set (fixed at each step).*

Remark 6. *It is worth mentioning that the method does not need to start at a fixed point of C, i.e. $x^0 \in C(x^0)$. Moreover, we do not assume that $x \in C(x)$ for all $x \in X$ as done in some existing works on extragradient algorithms for QEP's; see, for instance [49][Assumption A – c] and gap function; see [2].*

3.2 Convergence analysis

In this section, we present a convergence analysis of the proximal point method for solving quasi-equilibrium problems defined previously. Let $\{x^k\}$ be the sequence generated by Algorithm 5. We start the convergence analysis of $\{x^k\}$ checking that the stopping rule is practical.

Proposition 8. *If $x^{k+1} = x^k$ for some $k \in \mathbb{N}$, then x^k is a solution of QEP.*

Proof: From the definition of the algorithm, we have that $x^{k+1} \in S_{EP}(f_k, C_k)$. Thus,we have $x^{k+1} \in C_k$ and $f_k(x^{k+1}, y) \geqslant 0$ for all $y \in C_k$. Therefore

$$
f_k(x^{k+1}, y) = f(x^{k+1}, y) + \gamma_k \langle x^{k+1} - x^k, y - x^{k+1} \rangle \geqslant 0, \quad \forall y \in C_k. \tag{3.11}
$$

Since that $x^{k+1} = x^k$ we obtain that $\gamma_k\langle x^{k+1} - x^k, y - x^{k+1}\rangle = 0$. Follows from (3.11) that $f(x^{k+1}, y) \geqslant 0$, for all $y \in C_k$. Now, since that $x^{k+1} \in C_k$ and $x^{k+1} = x^k$, we get $x^k \in C_k$ and $f(x^k, y) \geqslant 0$, for all $y \in C_k$. Thus $x^k \in S_{QEP}(f, C)$ and the proof is finished. □

From now on, we assume that Algorithm 5 generates an infinite sequence $\{x^k\}$ in view of the last proposition.

Proposition 9. *Let $\{x^k\}$ be the sequence generated by the Algorithm 5. The following assertions hold:*

i) *$\{x^k\}$ is Fejér convergent to S^*;*

ii) *$\{x^k\}$ is bounded;*

iii) $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$.

Proof: We will test each item separately.

i) Take any $\tilde{x} \in S^* \subset S_{EP}(f, C(z))$, for all $z \in X$. Since that f is monotone, we get that $S_{EP}(f, C(z)) = S^d_{EP}(f, C(z))$, for all $z \in X$. In particular, for $z = x^k$ we have that

$$\tilde{x} \in S^d_{EP}(f, C_k), \quad \forall\, k \in \mathbb{N}.$$

Thus, applying Proposition 5 with $\widehat{f} = f_k$, $\widehat{x} = x^{k+1}$, and $x^* = x^k$ we obtain

$$\|x^{k+1} - \tilde{x}\|^2 + \|x^{k+1} - x^k\|^2 \leqslant \|x^k - \tilde{x}\|^2, \tag{3.12}$$

and therefore

$$\|x^{k+1} - \tilde{x}\| \leqslant \|x^k - \tilde{x}\|.$$

Since that $\tilde{x}$ was arbitrarily taken in S^*, we have that $\{x^k\}$ is Fejér convergent to S^*.

ii) $\{x^k\}$ is Féjer convergent, follows from lemma (2) that $\{x^k\}$ is bounded.

iii) Follows from (3.12),

$$0 \leqslant \|x^{k+1} - x^k\|^2 \leqslant \|x^k - \tilde{x}\|^2 - \|x^{k+1} - \tilde{x}\|^2. \tag{3.13}$$

Since the sequence of real numbers $\{\|x^k - \tilde{x}\|\}$ is non-increasing and bounded, we have that it is convergent. Thus, we have that

$$\lim_{k\to+\infty} \|x^k - \tilde{x}\| = L.$$

Therefore,

$$\lim_{k\to+\infty} \|x^{k+1} - \tilde{x}\| = L.$$

To the limit with $k \to +\infty$ in (3.13), we obtain

$$\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0.$$

□

Since the sequence is bound by the last proposition, next we prove that all the weak cluster points of $\{x^k\}$ are solution of the quasi-equilibrium problem.

Theorem 9. *Every cluster point of* $\{x^k\}$ *belongs to* $S_{QEP}(f, C)$*.*

Proof: Let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ that converges to $\widehat{x}$. From the definition of Algorithm 5 , we have that $x^{k_j+1} \in C(x^{k_j})$. From Proposition 9 (iii), we have

$$\lim_{j\to\infty} \|x^{k_j+1} - x^{k_j}\| = 0,$$

and hence, $\lim_{j\to\infty} x^{k_j+1} = \widehat{x}$. Thus, from the M-Continuity of C, we have that $\widehat{x} \in C(\widehat{x})$ and, given $y \in C(\widehat{x})$, there exists a sequence $\{y^{k_j}\}$ such that $y^{k_j} \to y$ and $y^{k_j} \in C(x^{k_j})$. Now, as $x^{k_j+1} \in S_{EP}(f_{k_j}, C_{k_j})$ we have

$$0 \leqslant f_{k_j}(x^{k_j+1}, z), \quad \forall z \in C(x^{k_j}),$$

which means in particular for $z = y^{k_j} \in C(x^{k_j})$ that

$$0 \leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\langle x^{k_j+1} - x^{k_j}, y^{k_j} - x^{k_j+1}\rangle \quad \forall j \in \mathbb{N}$$

Using the Cauchy–Schwartz inequality, we have,

$$0 \leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\|x^{k_j+1} - x^{k_j}\|\|y^{k_j} - x^{k_j+1}\| \quad \forall j \in \mathbb{N}.$$

Using the fact that $\{\gamma_{k_j}\}$, $\{x^{k_j}\}$ and $\{y^{k_j}\}$ are bounded sequences, f satisfies $ii)$(assumptions 1) and taking the limit as $j \to \infty$ in the last inequality, we have

$$0 \leqslant f(\widehat{x}, y).$$

Since we consider an arbitrary $y \in C(\widehat{x})$ this means that $0 \leqslant f(\widehat{x}, y)$, for all $y \in C(\widehat{x})$, and hence, $\widehat{x} \in S_{QEP}(f, C)$. This completes the proof.

□

Note that $S^* \subset S_{QEP}(f, C)$ and from Proposition 9, we have that $\{x^k\}$ is Fejér convergent to S^*. Theorem 9 does not guarantee the weak cluster points of $\{x^k\}$ belong to S^*, so we cannot apply Lemma 2 in order to obtain weak convergence of the whole sequence to a point of $S_{QEP}(f, C)$. Next result gives a sufficient condition to overcome this drawback.

Corollary. 1. *If* f *is strictly monotone, then* $\{x^k\}$ *weakly converges to a solution of* QEP.

Proof: We begin our proof with the following statements:

Statement 1: If f is strictly monotone, then $S_{QEP}(f, C) = S^* = \{x^*\}$, where x^* is a weak accumulation point of $\{x^k\}$. Once this fact is true, we get the result because:

1) By the proposition 9 we have that $\{x^k\}$ is Fejér Convergent, bounded, and $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$.

2) Every accumulation point of $\{x^k\} \in S_{QEP}(f, C)$. Since $\{x^k\}$ is bounded, it admits a convergent subsequence. As it has a single point of accumulation, the whole sequence is convergent.

Statement 2 : $S^* \subset S_{QEP}(f, C)$

Let $x^* \in S^* \Rightarrow x^* \in \{\cap_{z\in X} C(z); f(x^*, y) \geqslant 0, \quad \forall y \in \cup_{z\in X} C(z)\}$. Note that, as $x^* \in \cap_{z\in X} C(z) \quad \forall z \in X$, we have in particular that $x^* \in C(x^*)$. Furthermore, since that $f(x^*, y) \geqslant 0, \quad \forall\, y \in \cup_{z\in X} C(z)$, we have $f(x^*, y) \geqslant 0, \quad \forall\, y \in C(x^*)$. Thus proving our second statement. Finally, let $\widehat{x} \in S_{QEP}(f, C)$ and $x^* \in S^*$ be arbitrary points. Then:

$$f(\widehat{x}, y) \geqslant 0, \quad \forall\, y \in C(\widehat{x}) \tag{3.14}$$

e $x^* \in \{\cap_{z\in X} C(z)\}$.

$$f(x^*, y) \geqslant 0, \quad \forall y \in \bigcup_{z\in X} C(z). \tag{3.15}$$

Setting $y = x^*$ in (3.14), we get

$$f(\widehat{x}, x^*) \geqslant 0.$$

On the other hand, making $y = \widehat{x}$ in (3.15), and using the fact that f is monotonic comes that $f(x^*, \widehat{x}) \leqslant 0$. Since $f(x^*, \widehat{x}) \geqslant 0$, we conclude that

$$f(\widehat{x}, x^*) = 0. \tag{3.16}$$

Now, as $x^* \in S^* \subset S_{QEP}(f, C)$, we have $f(x^*, \widehat{x}) \geqslant 0$. If it holds that $x^* \neq \widehat{x}$, we have the strict monotonicity of f that

$$0 \leqslant f(x^*, \widehat{x}) = f(x^*, \widehat{x}) + f(\widehat{x}, x^*) < 0. \tag{3.17}$$

Therefore, we must have $x^* = \widehat{x}$ and consequently $S^* = S_{QEP}(f, C) = \{x^*\}$. □

Chapter 4

Quasi-Equilibrium Problems:Exact and Inexact Versions

In this chapter, we present the results we obtained in our studies. We started the chapter by proposing a version of the proximal point method with the Bregman distance in space $\mathbb{R}^n$. Posteriorly, we propose an inexact version of the method proposed by Santos and Souza [47] in Hilbert Space. In both methods, we perform a convergence analysis, obtaining the classic results on the convergence of sequences generated in proximal point algorithms.

4.1 Bregman regularized

In this section we state a quasi-equilibrium generalization of the proximal point method proposed by Burachik and Kassay [4] in the space $\mathbb{R}^n$.

Algorithm 6 : Bregman Proximal Point Method

1: Take a bounded sequence of positive parameters $\{\gamma_k\}$, choose $x^0 \in X$ and set $k = 0$

2: Given x^k, compute $x^{k+1} \in S_{EP}(f_k, C_k)$, where

$$f_k(x, y) = f(x, y) + \gamma_k \langle \nabla\varphi(x) - \nabla\varphi(x^k), y - x\rangle \tag{4.1}$$

and $C_k := C(x^k)$.

3: If $x^{k+1} = x^k$, stop and return x^k. Otherwise, set $k = k + 1$ and return to previous step.

Remark 7. *Note that Algorithm 6 solves, at each iteration, a Bregman regularized equilibrium problem. Thus, the well-definition of the method depends on* (4.1) *has a solution. In [4, Corollary 3.2], it is proved that* (4.1) *has a solution if the following assumption holds: Given* $\bar{x} \in X$ *fixed, if for every sequence* $\{x^k\} \subset X$ *such that* $\lim_{k\to\infty} \|x^k\| = \infty$, *we have*

$$\liminf_{k\to\infty} \left(f(\bar{x}, x^k) + \gamma \langle \nabla\varphi(\bar{x}) - \nabla\varphi(x^k), \bar{x} - x^k \rangle \right) > 0. \tag{4.2}$$

Moreover, if φ *is strictly convex, then* (4.1) *has a unique solution. In [4, Remark 3.1], it is shown that the above condition is weaker than to suppose that the Bregman function* φ *is coercive, i.e.,*

$$\lim_{\|x\|\to\infty} \frac{\varphi(x)}{\|x\|} = +\infty, \tag{4.3}$$

see [44, Lemma 1] and [4, Corollary 3.3]. On the other hand, as mentioned by Censor et al. [8, page 380], if φ *is a Bregman function with zone* S *and* $S' \subset S$ *is convex and closed, then* φ *can also be considered a Bregman function with zone* S'. *This fact can be applied to Algorithm 6 taking into account that* $C_k \subset X$ *is convex and closed, for all* $k \in \mathbb{N}$, *together with the assumptions made on the Bregman distance* D_φ *with zone* X *and the bifunction* f. *Therefore, one can ensure that Step 2 of Algorithm 6 is well-defined (i.e.,* (4.1) *has a solution) by assuming that* (4.2) *(or alternatively* (4.3)*) holds.*

Next, we show that if Algorithm 6 stops at iterate x^k, then this point is a solution of the QEP.

Proposition 10. *If* $x^{k+1} = x^k$ *for some* $k \in \mathbb{N}$, *then* x^k *is a solution of QEP.*

Proof: From the definition of the algorithm, we have that $x^{k+1} \in S_{EP}(f_k, C_k)$. Thus, we have $x^{k+1} \in C_k$ and $f_k(x^{k+1}, y) \geqslant 0$ for all $y \in C_k$. Therefore

$$f_k(x^{k+1}, y) = f(x^{k+1}, y) + \gamma_k \langle \nabla\varphi(x^{k+1}) - \nabla\varphi(x^k), y - x^{k+1} \rangle \geqslant 0, \quad \forall y \in C_k. \tag{4.4}$$

Since $x^{k+1} = x^k$ we obtain that $\gamma_k \langle \nabla\varphi(x^{k+1}) - \nabla\varphi(x^k), y - x^{k+1} \rangle = 0$. Follows from (4.4) that $f(x^{k+1}, y) \geqslant 0$, for all $y \in C_k$. Now, since that $x^{k+1} \in C_k$ and $x^{k+1} = x^k$, we get $x^k \in C_k$ and $f(x^k, y) \geqslant 0$, for all $y \in C_k$. Thus $x^k \in S_{QEP}(f, C)$ and the proof is finished. □

In the next result, let us state and prove a Bregman version of [47, proposition 2.5] that we will use in the next results. To this end, let $\bar{x}$ be fixed, define

$$\tilde{f}(x, y) = f(x, y) + \gamma \langle \nabla\varphi(x) - \nabla\varphi(\bar{x}), y - x \rangle, \tag{4.5}$$

for some $\gamma > 0$.

Proposition 11. *Let* $\bar{x} \in X$ *be an arbitrary point,* $\tilde{x}, x^* \in X$ *such that* $\tilde{x} \in S_{EP}(\tilde{f}, C(\bar{x}))$ *and* $x^* \in S^d_{EP}(f, C(\bar{x}))$*. If* f *satisfies Assumption 2:* $i)$-$iii)$*, then*

$$D_\varphi(x^*, \tilde{x}) + D_\varphi(\tilde{x}, \bar{x}) \leqslant D_\varphi(x^*, \bar{x}).$$

Proof: Since that $\tilde{x} \in S_{EP}(\tilde{f}, C(\bar{x}))$, we have that $\tilde{f}(\tilde{x}, y) \geqslant 0$, for all $y \in C(\bar{x})$. This means that

$$0 \leqslant f(\tilde{x}, y) + \gamma\langle \nabla\varphi(\tilde{x}) - \nabla\varphi(\bar{x}), y - \tilde{x}\rangle, \quad \forall\, y \in C(\bar{x}). \tag{4.6}$$

Now, as $x^* \in S^d_{EP}(f, C(\bar{x}))$, we have that $x^* \in C(\bar{x})$ and, in addition, $f(x, x^*) \leqslant 0$, for all $x \in C(\bar{x})$. In particular, $f(\tilde{x}, x^*) \leqslant 0$. Making $y = x^*$ in (4.6) together with $\gamma > 0$, we obtain

$$0 \leqslant \langle \nabla\varphi(\tilde{x}) - \nabla\varphi(\bar{x}), x^* - \tilde{x}\rangle.$$

Finally, from (1.3), we have

$$0 \leqslant \langle \nabla\varphi(\tilde{x}) - \nabla\varphi(\bar{x}), x^* - \tilde{x}\rangle = D_\varphi(x^*, \bar{x}) - D_\varphi(x^*, \tilde{x}) - D_\varphi(\tilde{x}, \bar{x}).$$

Consequently,

$$D_\varphi(x^*, \tilde{x}) + D_\varphi(\tilde{x}, \bar{x}) \leqslant D_\varphi(x^*, \bar{x}).$$

□

4.1.1 Convergence Analysis

Next, we prove some classical properties of the proximal point method for the sequence $\{x^k\}$ generated by the Algorithm 6.

Theorem 10. *Consider* $\{x^k\}$ *the sequence generated by algorithm 6. Then*

i) $\{x^k\}$ *is bounded;*

ii) $\lim_{k\to+\infty} D_\varphi(x^{k+1}, x^k) = 0$.

Proof: For the proof of i),let $x^* \in S^*$ be arbitrary. Since $S^* \subset S_{EP}(f, C(z))$, for all $z \in X$, we have that $x^* \in S_{EP}(f, C(z))$ and hence $f(x^*, y) \geqslant 0$, for all $y \in C(z)$. Now, since that f is monotone, we have $f(y, x^*) \leqslant 0$, for all $y \in C(z)$. This implies that $x^* \in S^d_{EP}(f, C(z))$, for all $z \in X$, and, in particular, for $z = x^k$. From the definition of

Algorithm 6, we have $x^{k+1} \in S_{EP}(f_k, C(x^k))$. Thus, applying Proposition 11 with $\tilde{f} = f_k$ in (4.1), $\tilde{x} = x^{k+1}, \bar{x} = x^k$ we have that

$$D_\varphi(x^*, x^{k+1}) + D_\varphi(x^{k+1}, x^k) \leqslant D_\varphi(x^*, x^k), \quad k \in \mathbb{N}. \tag{4.7}$$

Since $D_\varphi(x^{k+1}, x^k) \geqslant 0$, we have

$$D_\varphi(x^*, x^{k+1}) \leqslant D_\varphi(x^*, x^k), \quad \forall k \in \mathbb{N}, \quad x^* \in S^*.$$

It follows from the last inequality that $\{D_\varphi(x^*, x^k)\}$ is non-increasing, and since it is non-negative, it converges. In particular, it is bounded. Thus, the first assertion directly follows from condition $i)$ in Assumption 2.

Now, We will the prove the item ii). From (4.7), we have

$$D_\varphi(x^*, x^{k+1}) \leqslant D_\varphi(x^*, x^{k+1}) + D_\varphi(x^{k+1}, x^k) \leqslant D_\varphi(x^*, x^k), \quad \forall k \in \mathbb{N}.$$

Letting $k \to \infty$ in the last inequality and taking into account that $\{D_\varphi(x^*, x^k)\}$ is convergent,i.e. $\lim_{k\to+\infty} D_\varphi(x^*, x^k) = L$. So

$$\lim_{k\to+\infty} D_\varphi(x^*, x^{k+1}) \leqslant \lim_{k\to+\infty} D_\varphi(x^*, x^{k+1}) + \lim_{k\to+\infty} D_\varphi(x^{k+1}, x^k) \leqslant \lim_{k\to+\infty} D_\varphi(x^*, x^k),$$

consequently,

$$L \leqslant L + \lim_{k\to+\infty} D_\varphi(x^{k+1}, x^k) \leqslant L.$$

Therefore, $\lim_{k\to+\infty} D_\varphi(x^{k+1}, x^k) = 0$. □

Next, we prove our main convergence result.

Theorem 11. *Every weak cluster point of* $\{x^k\}$ *belongs to* $S_{QEP}(f, C)$.

Proof: It follows from the last theorem i) that $\{x^k\}$ is bounded. Let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ that converges to $\widehat{x}$. From the definition of Algorithm 6, we have that $x^{k_j+1} \in C(x^{k_j})$. It follows from Theorem 10 ii), we have

$$\lim_{k\to\infty} D_\varphi(x^{k+1}, x^k) = 0,$$

and hence, we can guarantee from condition $ii)$ in Assumption 2 that $\lim_{j\to\infty} x^{k_j+1} = \widehat{x}$. Thus, from the M-Continuity of C, we have that $\widehat{x} \in C(\widehat{x})$ and, given $y \in C(\widehat{x})$, there exists a sequence $\{y^{k_j}\}$ such that $y^{k_j} \to y$ and $y^{k_j} \in C(x^{k_j})$. Now, as $x^{k_j+1} \in S_{EP}(f_{k_j}, C_k)$ we have

$$f_{k_j}(x^{k_j+1}, z) \geqslant 0, \quad \forall z \in C(x^{k_j}),$$

which means in particular for $z = y^{k_j} \in C(x^{k_j})$ that

$$f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\langle \nabla\varphi(x^{k_j+1}) - \nabla\varphi(x^{k_j}), y^{k_j} - x^{k_j+1}\rangle \geqslant 0.$$

Using the Cauchy–Schwartz inequality, we have

$$f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\|\nabla\varphi(x^{k_j+1}) - \nabla\varphi(x^{k_j})\|\|y^{k_j} - x^{k_j+1}\| \geqslant 0.$$

Using the fact that $\{\gamma_{k_j}\}$, $\{x^{k_j}\}$ and $\{y^{k_j}\}$ are bounded sequences, φ is continuously differentiable, f satisfies $ii)$ in Assumptions 2 and taking the limit as $j \to \infty$ in the last inequality, we have

$$f(\widehat{x}, y) \geqslant 0.$$

Since that we consider an arbitrary $y \in C(\widehat{x})$ this means that $f(\widehat{x}, y) \geqslant 0$, for all $y \in C(\widehat{x})$, and hence, $\widehat{x} \in S_{QEP}(f, C)$. This completes the proof. □

4.1.2 Numerical Experiments

In this section, we illustrate the performance of the proposed method on one test problem adapted from [47] . We compare the performance of two Bregman regularized versions with the (classical) proximal point method for quasi-equilibrium problems proposed by [47]. We refrain from discussing computational efficiency of other methods, and hence, we skip discussions of comparisons of the proposed methods with other methods for QEP's.

The algorithms are coded in MATLAB R2020b on a 8 GB RAM Intel Core i7 to obtain the numerical results. The stopping rule is $\|x^{k+1} - x^k\| < 10^{-5}$. We take $\gamma_k = \gamma = 3.5$, for all $k \in \mathbb{N}$. We solve the subproblem (4.1) by using the regularized method in Muu and Quoc [35] in the classical version and the Bregman regularized method in Flam and Antipin [16]. They consider the following iterative method for solving an equilibrium problem: for any starting point $x^0 \in X$ and $\gamma > 0$, given $x^k \in X$ define $x^{k+1} \in X$ such that

$$x^{k+1} = \arg\min_{y \in X}\{\gamma f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\} \tag{4.8}$$

and

$$x^{k+1} = \arg\min_{y \in X}\{\gamma f(x^k, y) + D_\varphi(y, x^k)\}, \tag{4.9}$$

respectively. The solutions of the subproblems in (4.36) and (4.9) are computed by the build-in MATLAB solver “fmincon".

Example 4. *[47, Example 4.1 - Adapted] Consider the 2-dimensional nonsmooth quasi-equilibrium problem with the bifunction* $f : X \times X \to \mathbb{R}$ *given by*

$$f(x, y) = | y_1 | - | x_1 | + y_2^2 - x_2^2$$

and the multivalued mapping C *given by*

$$C(x) = \left\{ y \in X \,;\, y_1 + y_2 = 1 + \frac{| x_1 |}{1 + | x_1 |},\ y_i \geqslant 0.1 \quad i = 1, 2 \right\},$$

where $X \subset \mathbb{R}^2_{++}$ *is given by* $X = [0.1, +\infty) \times [0.1, +\infty)$. *One can check that* f *is monotone and the solution set is the single point* $x^* = (1, \frac{1}{2})$.

We run Algorithm 5 with 100 random starting points in the box $[0.1,\ 20] \times [0.1,\ 20]$. We compare the performance of the method with the Bregman functions given in Example 8 (called PPM), Example 9 (called BPPM-1) and Example 10 (called BPPM-2). At each running, the methods start from the same initial point and use the same constant γ. In the Tables 4.1,4.2 and 4.3, we show the results of all methods in terms of number of iterates, CPU time and the accuracy $\Upsilon(x^*)$ (cf. Proposition 6) until the stopping rule is satisfied, respectively. In these tables, min. iter. (resp. min. time), max. iter. (resp. max. time) and med. iter. (resp. med. time) denote the minimal, maximum and median of iterates (resp. CPU time) in 100 runs of the methods as well as min. $\Upsilon(x^*)$, max. $\Upsilon(x^*)$ and med. $\Upsilon(x^*)$ stand respectively to the minimum, maximum and median of the values $\Upsilon(x^*)$ in 100 runs, where x^* is the solution found by the methods.

As we can see in Tables 4.1 and 4.2, the Bregman regularized methods outperform the classical proximal point method in both number of iterates and CPU time. Furthermore, as shown in Table 4.3 all the methods find a good approximation of the solution.

Table 4.1: Running 100 times Algorithm 5 for Example 4.

Algorithm	min. iter.(k)	max. iter.(k)	med. iter.(k)
PPM	7	14	12.65
BPPM-1	8	17	11.69
BPPM-2	7	15	12.25

In Figures 4.4,4.5 and 4.6, we consider a particular instance of each method (using the same random starting point $x^0 = (8.493, 18.3231)$) to illustrate the assertions in

Table 4.2: Running 100 times Algorithm 5 for Example 4.

Algorithm	min. CPU time	max. CPU time	med. CPU time
PPM	0.0992826	1.1111657	0.199839135
BPPM-1	0.0840165	0.2752377	0.171949707
BPPM-2	0.0973115	0.4178743	0.178682051

Table 4.3: Running 100 times Algorithm 5 for Example 4.

Algorithm	min. $\Upsilon(x^*)$	max. $\Upsilon(x^*)$	med. $\Upsilon(x^*)$
PPM	0	2.360499186405201e-06	4.720998372810402e-08
BPPM-1	0	2.880658707330110e-06	5.761317414660219e-08
BPPM-2	0	2.687690141983094e-06	5.375380283966188e-08

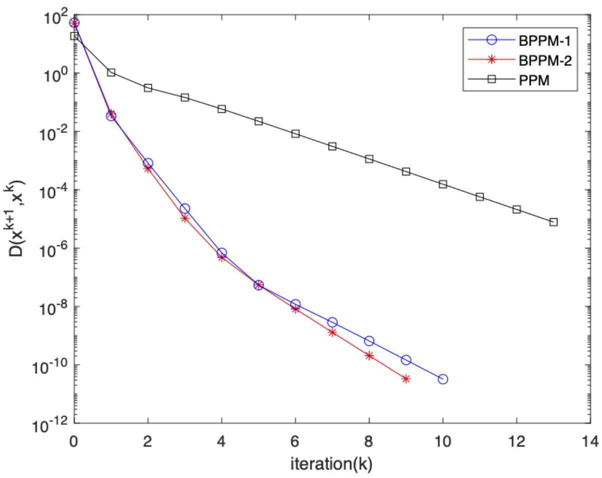


Figure 4.1: Behavior of $\{D_\varphi(x^{k+1}, x^k)\}$ and $\{||x^{k+1} - x^k||\}$.

Theorems 10 and 11. Figure 4.4 shows that the sequence $\{D_\varphi(x^{k+1}, x^k)\}$ converges to zero faster than $\{||x^{k+1} - x^k||\}$ using both Bregman distances in Example 9 and 10. In Figures 4.5 and 4.6 we can see that the sequence $\{x^k\}$ generated by the methods BPPM-1 and BPPM-2 approach the solution of the QEP faster than the Euclidean regularized PPM.

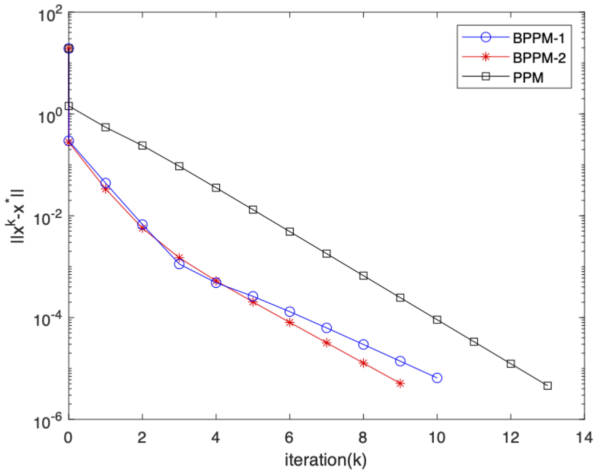


Figure 4.2: Behavior of $\{\|x^k - x^*\|\}$.

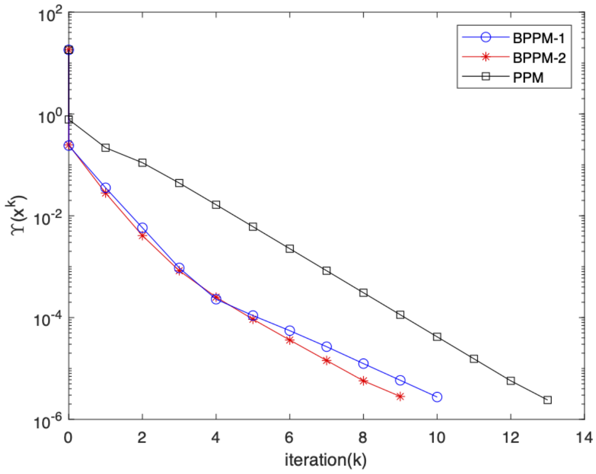


Figure 4.3: Behavior of $\Upsilon(x^k)$.

4.2 Quasi-Equilibrium Problems: Inexact Versions

In this section we state two inexact versions of the proximal point method for Hilbert space. The versions proposed here are generalizations of the work done by Santos and Souza [47] and Konnov [29].

Let $\bar{x} \in X$, $e \in H^*$ and $\gamma > 0$ be arbitrary. Given an equilibrium bifunction f, consider the following inexactly regularized equilibrium bifunction:

$$\tilde{f}(x, y) = f(x, y) + \gamma\langle x - \bar{x}, y - x\rangle - \langle e, y - x\rangle. \tag{4.10}$$

Proposition 12. *Let $x^*, \hat{x} \in C(\bar{x})$ be such that $\hat{x} \in S_{EP}(\tilde{f}, C(\bar{x}))$ and $x^* \in S^d(f, C(\bar{x}))$,* *where* $\tilde{f}$ *is given by* (4.10). *If* f *satisfies Assumption 1, then*

$$\|\bar{x}-\hat{x}\|^2 + \|\hat{x}-x^*\|^2 \leqslant \|\bar{x}-x^*\|^2 - \frac{2}{\gamma}\langle e, x^*-\hat{x}\rangle. \tag{4.11}$$

Moreover, if $\|x^*-\hat{x}-e\| \leqslant \max\{\|e\|, \|x^*-\hat{x}\|\}$*, then*

$$\|\bar{x}-\hat{x}\|^2 + \|\hat{x}-x^*\|^2 \leqslant \|\bar{x}-x^*\|^2. \tag{4.12}$$

Proof: Since that $\hat{x} \in S_{EP}(\tilde{f}, C(\bar{x}))$, we have that $\hat{x} \in C(\bar{x})$ and

$$\tilde{f}(\hat{x}, y) \geqslant 0, \quad \forall\, y \in C(\bar{x}).$$

This means that

$$0 \leqslant f(\hat{x}, y) + \gamma\langle \hat{x}-\bar{x}, y-\hat{x}\rangle - \langle e, y-\hat{x}\rangle, \quad \forall y \in C(\bar{x}). \tag{4.13}$$

Now, since that $x^* \in S^d_{EP}(f, C(\bar{x}))$, we have that $x^* \in C(\bar{x})$ and

$$f(x, x^*) \leqslant 0, \quad \forall x \in C(\bar{x}).$$

Taking $x = \hat{x}$ in the last inequality and using this fact into (4.13) with $y = x^*$, we obtain

$$0 \leqslant \gamma\langle \hat{x}-\bar{x}, x^*-\hat{x}\rangle - \langle e, x^*-\hat{x}\rangle. \tag{4.14}$$

Since that

$$\langle \hat{x}-\bar{x}, x^*-\hat{x}\rangle = \frac{1}{2}\left(\|x^*-\bar{x}\|^2 - \|\hat{x}-\bar{x}\|^2 - \|\hat{x}-x^*\|^2\right),$$

we have, from (4.14), that

$$\begin{aligned} 0 &\leqslant \gamma\langle \hat{x}-\bar{x}, x^*-\hat{x}\rangle - \langle e, x^*-\hat{x}\rangle \\ &= \frac{\gamma}{2}\left(\|x^*-\bar{x}\|^2 - \|\hat{x}-\bar{x}\|^2 - \|\hat{x}-x^*\|^2\right) - \langle e, x^*-\hat{x}\rangle, \end{aligned}$$

and using the fact that $\gamma > 0$, we prove the first assertion. To prove the second assertion, first note that

$$\langle e, x^*-\hat{x}\rangle = \frac{1}{2}\left(\|x^*-\hat{x}\|^2 + \|e\|^2 - \|x^*-\hat{x}-e\|^2\right) \geqslant 0, \tag{4.15}$$

where the inequality comes from the assumption $\|x^*-\hat{x}-e\| \leqslant \max\{\|e\|, \|x^*-\hat{x}\|\}$. Therefore, using (4.15) in (4.11) together with the fact that $\gamma > 0$, we obtain

$$\begin{aligned} 0 &\leqslant \frac{\gamma}{2}\left(\|x^*-\bar{x}\|^2 - \|\hat{x}-\bar{x}\|^2 - \|\hat{x}-x^*\|^2\right) - \langle e, x^*-\hat{x}\rangle \\ &\leqslant \|x^*-\bar{x}\|^2 - \|\hat{x}-\bar{x}\|^2 - \|\hat{x}-x^*\|^2. \end{aligned}$$

Therefore,

$$\|\hat{x}-\bar{x}\|^2+\|\hat{x}-x^*\|^2 \leqslant \|x^*-\bar{x}\|^2.$$

□

Next, we present two inexact versions of the proximal point method for solving a QEP. From now on, we assume that the equilibrium bifunction f satisfies i)-iii) in Assumption 1. Let S^* be the set given by

$$S^* = \left\{ x \in \bigcap_{z\in X} C(z) : f(x,y) \geqslant 0,\ \forall y \in \bigcup_{z\in X} C(z) \right\}. \tag{4.16}$$

This set will play an important role in our convergence analysis, and it has been considered in the convergence analysis of different algorithms in quasi-equilibrium problems; see [19], [30],[47],[49],[51],[53]. Note that $S^* \subset S_{QEP}(f,C)$, and hence, if $S^* \neq \emptyset$, then $S_{QEP}(f,C) \neq \emptyset$. Alternatively, we also consider the set S_* given by

$$S_* = \{x \in C(x) : f(x,y) \geqslant 0,\ \forall y \in X\}. \tag{4.17}$$

In this case, we also have that $S_* \subset S_{QEP}(f,C)$. Dual versions of the sets (4.16) and (4.17) were considered, for instance, in [30] and [10],[51] respectively.

Next, we provide some examples where the sets S^* and S_* are non-empty; see also [30, Example 1].

Example 5. *Consider $x^* \in S_{EP}(f,X)$ and let C be a point-to-set mapping given by*

$$C(x) = \{z \in X : \|z - x^*\| \leqslant \|x\|\},$$

i.e., for each $x \in X$, $C(x)$ is a closed ball center at x^* with ratio $\|x\|$. Clearly, $x^* \in \bigcap_{x\in X} C(x)$ and, in particular, $x^* \in C(x^*)$. Since $x^* \in S_{EP}(f,X)$, then we can easily see that $x^* \in S^*$ and $x^* \in S_*$. Therefore, the non-emptyness of S^* and S_* directly follows from their existence, which results in an equilibrium problem.

Example 6 (Minimization problem)**.** *Let $\varphi : X \subset \mathbb{R}^n \to \mathbb{R}$ be a function. The minimization problem*

$$\min_{x\in X} \varphi(x)$$

can be viewed as the quasi-equilibrium problem **QEP** *by taking*

$$C(x) = \{z \in X :\ \varphi(z) \leqslant \varphi(x)\} \quad \text{and} \quad f(x,y) = \varphi(y) - \varphi(x).$$

In this case, clearly $x \in C(x)$, *for every* $x \in X$, *and one can easily see that* $\arg\min_{x\in X} \varphi(x) = S^* = S_*$. *For that, be* $\bar{x} = \arg\min_{x\in X} \varphi(x)$ *then* $\bar{x} \in C(x)$ *for all* $x \in X$ *because*

$$\varphi(\bar{x}) \leqslant \varphi(x) \quad \forall x \in X.$$

Furthermore

$$f(\bar{x}, y) = \varphi(y) - \varphi(\bar{x}) \geqslant 0 \quad \forall y \in X.$$

Therefore, $\bar{x} \in S^*$ *and* $\arg\min_{x\in X} \varphi(x) \subset S^*$. *On the other hand, let* $x^* \in S^*$, *then*

$$x^* \in C(x) \quad \forall x \in X.$$

This means $\varphi(x^*) \leqslant \varphi(x)$ *for all* $x \in X$. *Therefore,* $x^* \in \arg\min_{x\in X} \varphi(x)$ *and* $S^* \subset \arg\min_{x\in X} \varphi(x)$. *In this way, it is proved* $S^* = \arg\min_{x\in X} \varphi(x)$. *For the second part, we will proved that* $S_* = \arg\min_{x\in X} \varphi(x)$. *Let* $\bar{x} \in \arg\min_{x\in X} \varphi(x)$, *then*

$$\bar{x} \in C(\bar{x}) \text{ and } f(\bar{x}, y) = \varphi(y) - \varphi(x) \geqslant 0 \quad \forall y \in X.$$

Therefore, $\bar{x} \in S_*$ *and* $\arg\min_{x\in X} \varphi(x) \subset S_*$. *Now, let* $\widehat{x} \in S_*$, *then*

$$\widehat{x} \in C(\widehat{x}) \text{ and } f(\widehat{x}, y) = \varphi(y) - \varphi(\widehat{x}) \geqslant 0.$$

So we have $\varphi(\widehat{x}) \leqslant \varphi(y)$ *for all* $y \in X$. *This means that* $\widehat{x} \in \arg\min_{x\in X} \varphi(x)$ *and* $S_* \subset \arg\min_{x\in X} \varphi(x)$. *So, it follows the equality* $S_* = \arg\min_{x\in X} \varphi(x)$. *Therefore, the non-emptyness of* S^* *and* S_* *depends on* $\arg\min_{x\in X} \varphi(x)$ *to be non-empty.*

Example 7 (Multi-objective problem). *Let* $\Phi : X \to \mathbb{R}^m$ *be a vector-valued function, i.e.,* $\Phi(x) = (\varphi_1(x), \ldots, \varphi_m(x))$, *where* $\varphi_i : X \to \mathbb{R}$ *is a scalar function for each* $i = 1, \ldots, m$. *The multi-objective optimization problem of finding Pareto and weak Pareto points of* Φ *is denoted by*

$$\mathbb{R}^m_+\text{-}\mathit{arg\,min}\{\Phi(x) \ : \ x \in X\} \quad \text{and} \quad \mathbb{R}^m_+\text{-}\mathit{arg\,min}_w\{\Phi(x) \ : \ x \in X\},$$

respectively, where a point $x^* \in X$ *is a Pareto point of* Φ *if there is no* $y \in X$ *such that* $\varphi_i(y) \leqslant \varphi_i(x^*)$, *for every* $i = 1, \ldots, m$, *with* $\Phi(y) \neq \Phi(x^*)$ *and a point* $x^* \in X$ *is a weak Pareto point of* Φ *if there is no* $y \in X$ *such that* $\varphi_i(y) < \varphi_i(x^*)$, *for all* $i = 1, \ldots, m$. *The problem of finding weak Pareto points of* Φ *can be viewed as the quasi-equilibrium problem* **QEP** *by taking*

$$C(x) = \{z \in X : \ \varphi_i(x) - \varphi_i(z) \geqslant 0, \ \text{for some } i = 1, \ldots, m\} \tag{4.18}$$

and

$$f(x,y) = \max_{1\leqslant i\leqslant m}\{\varphi_i(y) - \varphi_i(x)\}. \tag{4.19}$$

Note that $x \in C(x)$, *for every* $x \in X$, *and one can easily prove that* $\mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\} = S^* = S_*$. *To prove the previous equality, let* $\bar{x} \in \mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\}$, *then there is no* $y \in X$ *such that* $\varphi_i(y) < \varphi_i(\bar{x})$, *for every* $i = 1,\ldots,m$, *with* $\Phi(y) \neq \Phi(x^*)$. *This means that, there is* $p \in [1,\cdots,m]$ *such that* $\varphi_p(y) - \varphi_p(\bar{x}) \geqslant 0$. *Therefore,* $\bar{x} \in C(x)$ *for all* $x \in X$. *Furthermore*

$$f(\bar{x},y) = \max_{1\leqslant i\leqslant m}\{\varphi_i(y) - \varphi_i(\bar{x})\} \geqslant \varphi_p(y) - \varphi_p(\bar{x}) \geqslant 0.$$

Consequently, $\bar{x} \in S^*$ *and* $\mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\} \subset S^*$. *On the other hand, let* $x^* \in S^*$, *we have* $x^* \in C(x)$ *for all* $x \in X$. *Therefore,* $\varphi_i(x) - \varphi_i(x^*) \geqslant 0$, *for some* $i = 1,\cdots,m\}$. *This means that* $x^* \in \mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\}$ *and consequently* $S^* = \mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\}$. *For the proof of the second part, let* $\bar{x} \in \mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\}$, *then there is no* $y \in X$ *such that* $\varphi_i(y) < \varphi_i(\bar{x})$, *for every* $i = 1,\cdots,m$, *with* $\Phi(y) \neq \Phi(\bar{x})$. *This means that, there is* $j \in [1,\cdots,m]$ *such that* $\varphi_j(y) - \varphi_j(\bar{x}) \geqslant 0$. *Therefore,* $\bar{x} \in C(x)$ *for all* $x \in X$. *In particular,*$\bar{x} \in C(\bar{x})$. *Furthermore,*

$$f(\bar{x},y) = \max_{1\leqslant i\leqslant m}\{\varphi_i(y) - \varphi_i(\bar{x})\} \geqslant \varphi_j(y) - \varphi_j(\bar{x}) \geqslant 0. \quad \forall y \in X.$$

Consequently, $\mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\} \subset S_*$. *Now, let* $w \in S_*$, *then*

$$w \in C(w) \text{ and } f(w,y) = \max_{1\leqslant i\leqslant m}\{\varphi_i(y) - \varphi_i(w)\} \geqslant 0 \quad \forall y \in X.$$

This implies that there is $r \in [1,\cdots,m]$ *such that* $\varphi_r(y) - \varphi_r(w) \geqslant 0$. *Therefore* $w \in \mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\}$ *and finally* $S_* = \mathbb{R}^m_+$-*arg min*$_w\{\Phi(x) \,:\, x \in X\}$.

If we replace the mapping C *in* (4.18) *by*

$$C(x) = \{z \in X \,:\, \varphi_i(x) - \varphi_i(z) \geqslant 0, \text{ for some } i = 1,\cdots,m\}$$

and the bifunction f *in* (4.19) *by*

$$f(x,y) = \sum_{i=1}^{m}(\varphi_i(y) - \varphi_i(x)),$$

we have the problem of finding Pareto points of Φ *can be viewed as the quasi-equilibrium problem* **QEP** *and* $\mathbb{R}^m_+$-*arg min*$\{\Phi(x) \,:\, x \in X\} = S^* = S_*$. *To prove the previous equality, let* $\bar{z} \in \mathbb{R}^m_+$-*arg min*$\{\Phi(x) \,:\, x \in X\}$, *then there is no* $y \in X$ *such that* $\varphi_i(y) \leqslant \varphi_i(\bar{z})$, *for* *every* $i = 1, \ldots, m$, *with* $\Phi(y) \neq \Phi(\bar{z})$. *This means that, there is* $p \in [1, \cdots, m]$ *such that* $\varphi_p(y) - \varphi_p(\bar{z}) > 0$. *Therefore,* $\bar{z} \in C(x)$ *for all* $x \in X$. *Furthermore*

$$f(\bar{z}, y) = \sum_{i=1, i \neq p}^{m} (\varphi_i(y) - \varphi_i(\bar{z})) + (\varphi_p(y) - \varphi_p(\bar{z})) \geqslant \varphi_p(y) - \varphi_p(\bar{z} \geqslant 0.$$

Consequently, $\bar{z} \in S^*$ *and* $\mathbb{R}^m_+$*-arg min*$\{\Phi(x) \ : \ x \in X\} \subset S^*$. *On the other hand, let* $x^* \in S^*$, *we have* $x^* \in C(x)$ *for all* $x \in X$. *Therefore* $\varphi_i(x) - \varphi_i(x^*) \geqslant 0$, *for some* $i = 1, \ldots, m\}$. *This means that* $x^* \in \mathbb{R}^m_+$*-arg min*$\{\Phi(x) \ : \ x \in X\}$ *and consequently* $S^* = \mathbb{R}^m_+$*-arg min*$\{\Phi(x) \ : \ x \in X\}$. *For the proof of the second part, let* $\bar{z} \in \mathbb{R}^m_+$*-arg min*$\{\Phi(x) \ : \ x \in X\}$, *then there is no* $y \in X$ *such that* $\varphi_i(y) \leqslant \varphi_i(\bar{z})$, *for every* $i = 1, \ldots, m$, *with* $\Phi(y) \neq \Phi(\bar{z})$. *This means that, there is* $j \in [1, \cdots m]$ *such that* $\varphi_j(y) - \varphi_j(\bar{x}) \geqslant 0$. *Therefore* $\bar{z} \in C(x)$ *for all* $x \in X$. *In particular,* $\bar{z} \in C(\bar{z})$. *Furthermore,*

$$f(\bar{z}, y) = \sum_{i=1, i \neq j}^{m} (\varphi_i(y) - \varphi_i(\bar{z})) + (\varphi_j(y) - \varphi_j(\bar{z})) \geqslant \varphi_j(y) - \varphi_j(\bar{z}) \geqslant 0.$$

Consequently $\mathbb{R}^m_+$*-arg min*$\{\Phi(x) \ : \ x \in X\} \subset S_*$. *Now, let* $v \in S_*$, *then*

$$v \in C(v) \text{ and } f(v, y) = \sum_{i=1}^{m} (\varphi_i(y) - \varphi_i(v) \geqslant 0 \quad \forall y \in X.$$

This implies that there is $r \in [1, \ldots, m]$ *such that* $\varphi_r(y) - \varphi_r(v) \geqslant 0$. *Therefore,* $v \in \mathbb{R}^m_+$*-arg min*$\{\Phi(x) \ : \ x \in X\}$ *and finally* $S_* = \mathbb{R}^m_+$*-arg min*$\{\Phi(x) \ : \ x \in X\}$. *Therefore, the non-emptyness of the sets* S^* *and* S_* *depend on the non-emptyness of the solution sets* $\mathbb{R}^m_+$*-arg min*$\{\Phi(x) \ : \ x \in X\}$ *and* $\mathbb{R}^m_+$*-arg min*$_w\{\Phi(x) \ : \ x \in X\}$.

We will consider two kinds of erro criterior. In the first one, our method will solve the perturbed regularized equilibrium problem

$$f_k^e(x, y) = f_k(x, y) - \langle e^k, y - x \rangle \tag{4.20}$$

instead of solving the exact regularized equilibrium problem

$$f_k(x, y) = f(x, y) + \gamma_k \langle x - x^k, y - x \rangle.$$

In this case, we will suppose that at each iterate the error e^k committed is controlled, i.e., it satisfies one of the following conditions, for all $k \in \mathbb{N}$:

(E1) $\|e^k\| \leqslant \|x^{k+1} - x^k\|$;

Algorithm 7 : Inexact Proximal Point Method

1: Take a bounded auxiliary sequence of positive parameters $\{\gamma_k\}$ such that $0 < a \leqslant \gamma_k \leqslant b$, for all $k \in \mathbb{N}$, and choose $x^0 \in X$

2: Given $x^k \in X$, compute

$$x^{k+1} \in S_{EP}(f_k^e, C_k), \tag{4.21}$$

where $f_k^e(x, y) = f(x, y) + \gamma_k \langle x - x^k, y - x\rangle - \langle e^k, y - x\rangle$, $C_k = C(x^k)$ and the error $\{e^k\}$ satisfies (E1) or (E2)

3: If $x^{k+1} = x^k$, stop and return x^k. Otherwise, set $k = k + 1$ and retourn to step 2.

(E2) $\|x^{k+1} - x^k - e^k\| \leqslant \max\left\{\|x^{k+1} - x^k\|, \|e^k\|\right\}$.

This method is stated in Algorithm 7 in the sequel.

Remark 8. *Note that if $e^k = 0$, for all $k \in \mathbb{N}$, in (4.20), then Algorithm 7 becomes the proximal point method proposed by [47]. Since x^{k+1} is a solution to an equilibrium problem and from [24, Proposition 3.1] the set $S_{EP}(f_k^e, C_k)$ is non-empty, then Algorithm 7 is well-defined. It is worth mentioning that even in the EP setting, Algorithm 7 is not the same as the inexact algorithm proposed in [24, Algorithm I], where a projection step is performed after solving the inexactly regularized equilibrium problem. Furthermore, the assumptions (E1) and (E2) in the error $\{e^k\}$ are well known in the literature of inexact proximal point methods; see for instance [7][20],[41],[45],[48].*

In the second method, we will consider as the next iterate $x^{k+1} \in X$ a point close enough to the exact solution controlled by a summable sequence of errors $\{\varepsilon_k\}$. While the exact solution needs to find a point belonging to $C(x^k)$ which is a solution of the equilibrium problem with the regularized bifunction f_k, this inexact version takes as its next iteration any point in an ε-neighborhood (not necessarily in $C(x^k)$) of the exact solution of the subproblem. This method is stated in Algorithm 8 in the sequel.

Remark 9. *Clearly, if the sequence of error $\{\varepsilon_k\}$ is taken equal to zero, for all $k \in \mathbb{N}$, in Algorithm 8, then $x^{k+1} = v^k \in S_{EP}(f_k, C_k)$ is the exact solution of the subproblems, and hence, Algorithm 8 becomes the proximal point method proposed by [47]. The well-definition of Algorithm 8 comes from the well-definition of the exact method; see [47, Remark 3.1]. On the other hand, if $C_k = X$, for all $k \in \mathbb{N}$, then Algorithm 8 retrieves the inexact proximal point method proposed by Konnov [28] for solving equilibrium problems.*

Algorithm 8 : Inexact Proximal Point Method

1: Take a bounded auxiliary sequence of positive parameters $\{\gamma_k\}$ such that $0 < a \leqslant \gamma_k \leqslant b$, for all $k \in \mathbb{N}$, and choose $x^0 \in X$;

2: Given $\varepsilon_k \geqslant 0$, $x^k \in X$ and $v^k \in S_{EP}(f_k, C_k)$, compute $x^{k+1} \in X$ such that

$$\|x^{k+1} - v^k\| \leqslant \varepsilon_k \quad \text{with} \quad \sum_{k=0}^{\infty} \varepsilon_k < +\infty, \tag{4.22}$$

where $f_k(x, y) = f(x, y) + \gamma_k \langle x - x^k, y - x \rangle$ and $C_k = C(x^k)$.

The summable assumption on the error $\{\varepsilon_k\}$ *in* (4.22) *is standard, and it was considered, for instance, in [28][45][48].*

Remark 10. *See that, from the exact method, we know that the algorithm stops when* $v^k = v^{k-1}$. *Since* $\|x^{k+1} - v^k\| < \varepsilon_k$ *and* $\|x^k - v^{k-1}\| < \varepsilon_{k-1}$. *We verified euristically that assuming* $x^{k+1} = x^k$ *is a good stopping criterion for the algorithm.*

4.2.1 Convergence analysis

Next, we present the convergence results for the proposed methods. In the case of Algorithm 7, we will consider the error satisfying each one of the assumptions (E1) and (E2) separately. Let us start with Algorithm 7.

Proposition 13. *Let* $\{x^k\}$ *be the sequence generated by Algorithm 7. Additionally, suppose that the following conditions hold:*

$$\sum_{k=0}^{+\infty} \|e^k\| < +\infty \tag{4.23}$$

$$\sum_{k=0}^{+\infty} | \langle e^k, x^{k+1} \rangle | < +\infty. \tag{4.24}$$

One has,

(i) If $S^* \neq \emptyset$, *then* $\{x^k\}$ *is quasi-Fejér convergent to* S^*;

(ii) If $S_* \neq \emptyset$ *and* $S_* \subset C_k$, *for all* $k \in \mathbb{N}$, *then* $\{x^k\}$ *is quasi-Fejér convergent to* S_*;

Therefore, if item (i) or (ii) holds, then $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$.

Proof: To prove item (i) take $x^* \in S^*$. Since $S^* \subset S_{EP}(f, C(z))$, for all $z \in X$, we have that $x^* \in S_{EP}(f, C(z))$ and hence $f(x^*, y) \geqslant 0$, for all $y \in C(z)$. From $iii)$ in Assumption 1, f is monotone, then we have that $f(y, x^*) \leqslant 0$, for all $y \in C(z)$. This implies that $x^* \in S^d_{EP}(f, C(z))$, for all $z \in X$, in particular, for $z = x^k$. This means that $x^* \in S^d_{EP}(f, C_k)$. From the definition of the algorithm, we have that $x^{k+1} \in S_{EP}(f^e_k, C_k)$. Thus, applying Proposition 12 with $\tilde{f} = f^e_k$, $\tilde{x} = x^{k+1}, \bar{x} = x^k$ we have

$$\|x^{k+1} - x^*\|^2 + \|x^{k+1} - x^k\|^2 \leqslant \|x^k - x^*\|^2 - \frac{2}{\gamma_k}\langle e^k, x^* - x^{k+1}\rangle, \forall k \in \mathbb{N}. \tag{4.25}$$

This implies that

$$\|x^{k+1} - x^*\|^2 \leqslant \|x^k - x^*\|^2 - \frac{2}{\gamma_k}\langle e^k, x^* - x^{k+1}\rangle, \quad \forall k \in \mathbb{N},$$

and hence, we obtain

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leqslant \|x^k - x^*\|^2 + \frac{2}{\gamma_k}\left(|\langle e^k, x^{k+1}\rangle| + \|e^k\|\|x^*\|\right) \\ &\leqslant \|x^k - x^*\|^2 + \frac{2}{a}\left(|\langle e^k, x^{k+1}\rangle| + \|e^k\|\|x^*\|\right). \end{aligned}$$

Taking $\epsilon_k = \frac{2}{a}\left(|\langle e^k, x^{k+1}\rangle| + \|e^k\|\|x^*\|\right)$ from (4.23) and (4.24), we have that $\sum_{k=0}^{\infty} \varepsilon_k < \infty$. Thus, $\{x^k\}$ is quasi-Fejér convergent to S^* and the first assertion is proved.

The proof of item (ii) is quite similar to the previous one. If we prove that $x^* \in S_*$ implies that $x^* \in S^d_{EP}(f, C_k)$, for all $k \in \mathbb{N}$, thus the remainder of the proof is equal to the item (i) from (4.25) on. Indeed, if $x^* \in S_*$, then $x^* \in C(x^*) \subset X$ and $f(x^*, y) \geqslant 0$, for all $y \in X$. From the monotonicity of f, we have that $f(y, x^*) \leqslant 0$, for all $y \in X$, and hence, $x^* \in S^d_{EP}(f, X)$. Combining this fact with $S_* \subset C_k \subset X$, for all $k \in \mathbb{N}$, we have that $x^* \in S^d_{EP}(f, C_k)$ and the assertion is proved.

To prove the last assertion, first note that since $\{\gamma_k\}$ is bounded and assumptions (4.23) and (4.24) hold, we have that $\lim_{k\to+\infty} \frac{2}{\gamma_k}\langle e^k, x^* - x^{k+1}\rangle = 0$. Furthermore, from Lemma 3, we have that the sequence $\{\|x^k - x^*\|\}$ converges, for all $x^* \in S^*$ or $x^* \in S_*$. From (4.25), we obtain

$$\|x^{k+1} - x^*\|^2 \leqslant \|x^{k+1} - x^*\|^2 + \|x^{k+1} - x^k\|^2 \leqslant \|x^k - x^*\|^2 - \frac{2}{\gamma_k}\langle e^k, y - x^{k+1}\rangle.$$

Taking the limit as $k \to +\infty$ in the last inequality, we obtain $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$ and the proof is concluded. $\square$

Remark 11. *Assumptions* (4.23) *and* (4.24) *are well known for inexact proximal algorithms in different contexts; see, for instance, [7][12],[27][39]. As remarked by Eckstein [12] it may appear somewhat unnatural that* (4.24) *involves not only the error sequence but also the iterate* x^{k+1} *which cannot be determined before using the algorithm. However, the requirement* (4.24) *should not be too difficult to enforce in practice. For instance, one could stop the process of computing* x^{k+1} *as soon as* $\|e^k\| \leqslant L\alpha^k$ *and* $|\langle e^k, x^{k+1}\rangle| \leqslant \hat{L}\alpha^k$*, where* $L, \hat{L} \geqslant 0$ *and* $\alpha \in [0,1)$*. Furthermore, if one knows a priori that* $\{x^k\}$ *is bounded, then* (4.24) *is a simple consequence of* (4.23).

Remark 12. *Note that if* $\{x^k\}$ *is generated by Algorithm 7 with the error given by (E2), then Proposition 13 holds in the absence of the assumptions* (4.23) *and* (4.24)*. Indeed, from the second part of Proposition 12 with* $\tilde{f} = f_k^e$*,* $\tilde{x} = x^{k+1}$ *and* $\bar{x} = x^k$*, we have that*

$$\|x^{k+1} - x^*\|^2 + \|x^{k+1} - x^k\|^2 \leqslant \|x^k - x^*\|^2, \quad \forall k \in \mathbb{N}, \tag{4.26}$$

which implies that $\|x^{k+1} - x^*\| \leqslant \|x^k - x^*\|$*, for all* $k \in \mathbb{N}$*, and hence,* $\{x^k\}$ *is Fejér convergent to* S^**. Moreover, if the sequence* $\{\|x^k - x^*\|\}$ *is non-increasing and bounded from below, then it converges for all* $x^* \in S^*$*. Thus, Taking the limit as* $k \to +\infty$ *in* (4.26)*, we have that* $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$*. This remark is also true if we replace* S^* *by* S_**.*

Next, we show the main convergence result for Algorithm 7. To this end, we will use Proposition 13. In view of Remark 12, we will suppose that assumptions (4.23) and (4.24) hold only if the error $\{e^k\}$ satisfies (E1). Otherwise, if Algorithm 7 uses the error $\{e^k\}$ with (E2), we will replace assumptions (4.23) and (4.24) by the following:

$$\lim_{k\to\infty} \|e^k\| = 0. \tag{4.27}$$

Theorem 12. *Let* $\{x^k\}$ *be the sequence generated by Algorithm 7 and suppose that* S^* *or* S_* *is non-empty. In addition, if (E1) is used, we suppose the assumptions* (4.23) *and* (4.24) *hold. Otherwise, if (E2) is used, we assume that* (4.27) *holds. Then, every weak cluster point of* $\{x^k\}$ *is a solution to the* $\mathsf{QEP}(f, C)$*.*

Proof: If S^* is non-empty (respectively, S_* is non-empty) from Proposition 13 (i) (respectively, Proposition 13 (ii)), we have that $\{x^k\}$ is quasi-Fejér convergent to S^* (respectively, S_*). Thus, from Lemma 2, $\{x^k\}$ is bounded. Let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ weakly converging to x^*. From the definition of the algorithm, we have that $x^{k_j+1} \in C(x^{k_j})$ and hence, from the M-continuity of C, we obtain that $x^* \in C(x^*)$ taking into account that by Proposition 13 (ii), $x^{k_j+1} \rightharpoonup x^*$.

Now, for any $y \in C(x^*)$, again from the M-continuity of C, there exists a sequence $\{y^{k_j}\}$ such that $y^{k_j} \to y$ and $y^{k_j} \in C(x^{k_j})$. Since $x^{k_j+1} \in S_{EP}(f^e_{k_j}, C_{k_j})$, then

$$f^e_{k_j}(x^{k_j+1}, z) \geqslant 0, \quad \forall z \in C(x^{k_j}),$$

which means that

$$0 \leqslant f(x^{k_j+1}, z) + \gamma_{k_j}\langle x^{k_j+1} - x^{k_j}, z - x^{k_j+1}\rangle - \langle e^{k_j}, z - x^{k_j+1}\rangle, \quad \forall z \in C(x^{k_j}). \tag{4.28}$$

We prove that the result holds using the two criteria $(E1)$ and $(E2)$ for the error $\{e^k\}$. First, let us consider Algorithm 7 with the error $\{e^k\}$ satisfying condition $(E1)$. From (4.28) with $z = y^{k_j}$ we have

$$\begin{aligned} 0 &\leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\langle x^{k_j+1} - x^{k_j}, y^{k_j} - x^{k_j+1}\rangle - \langle e^{k_j}, y^{k_j} - x^{k_j+1}\rangle \\ &\leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\|x^{k_j+1} - x^{k_j}\| \, \|y^{k_j} - x^{k_j+1}\| + \|e^{k_j}\| \, \|x^{k_j+1} - y^{k_j}\| \\ &\leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\|x^{k_j+1} - x^{k_j}\| \, \|y^{k_j} - x^{k_j+1}\| + \|x^{k_j+1} - x^{k_j}\| \, \|x^{k_j+1} - y^{k_j}\|, \end{aligned}$$

where the second inequality comes from the Cauchy-Schwarz inequality and the last inequality was applied $(E1)$. Taking the limit as $j \to +\infty$ in the last inequality and taking into account that the sequences $\{\gamma_k\}$, $\{x^k\}$ and $\{y^k\}$ are bounded, Proposition 13 (ii) and i) in Assumption 1, we obtain

$$f(x^*, y) \geqslant 0, \quad \forall y \in C(x^*).$$

This means that $x^* \in S_{QEP}(f, C)$ and the first part of the theorem is proved.

Let us now assume that Algorithm 7 uses the error $\{e^k\}$ satisfying condition $(E2)$. Rewriting (4.28) with $z = y^{k_j}$, we have

$$\begin{aligned} 0 &\leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\langle x^{k_j+1} - x^{k_j} - e^{k_j}, y^{k_j} - x^{k_j+1}\rangle \\ &\leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\|x^{k_j+1} - x^{k_j} - e^{k_j}\| \, \|y^{k_j} - x^{k_j+1}\|. \end{aligned} \tag{4.29}$$

From $(E2)$, if $\|x^{k_j+1} - x^{k_j} - e^{k_j}\| \leqslant \|x^{k_j+1} - x^{k_j}\|$, then (4.29) implies

$$0 \leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\|x^{k_j+1} - x^{k_j}\| \, \|y^{k_j} - x^{k_j+1}\|. \tag{4.30}$$

Otherwise, if $\|x^{k_j+1} - x^{k_j} - e^{k_j}\| \leqslant \|e^{k_j}\|$, then (4.29) implies

$$0 \leqslant f(x^{k_j+1}, y^{k_j}) + \gamma_{k_j}\|e^{k_j}\| \, \|y^{k_j} - x^{k_j+1}\|. \tag{4.31}$$

Taking the limit as $j \to +\infty$, in (4.30) and (4.31), and taking into account that the sequences $\{\gamma_k\}$, $\{x^k\}$ and $\{y^k\}$ are bounded, from Proposition 13 (ii) $\lim_{j\to+\infty} \|x^{k_j+1} - x^{k_j}\| = 0$ and $\lim_{k\to\infty} \|e^k\| = 0$ (see (4.27) in Remark 12) and $i)$ in Assumption 1, we have

$$f(x^*, y) \geqslant 0, \quad \forall y \in C(x^*),$$

having in mind that C is M-continuous. This means that $x^* \in S_{QEP}(f, C)$ and the second part of the theorem are proved. □

Now, we show the convergence results for Algorithm 8. In contrast with Algorithm 7, here we will consider only the case where S^* is non-empty.

Proposition 14. *Let $\{x^k\}$ be the sequence generated by Algorithm 8. Then, the following assertions hold:*

i) $\{x^k\}$ *is quasi-Fejér convergent to* S^**;*

ii) $\lim_{k\to+\infty} \|x^{k+1} - x^k\| = 0$ *and* $\lim_{k\to+\infty} \|x^k - v^k\| = 0$.

Proof: (i): Take any $x^* \in S^*$. Thus, from $iii)$ in Assumption 1, we have that $f(v^k, x^*) \leqslant 0$, for all $k \in \mathbb{N}$. Now, from the definition of Algorithm 8, we have that $v^k \in S_{EP}(f_k, C_k)$, i.e.,

$$f(v^k, y) + \gamma_k\langle v^k - x^k, y - v^k\rangle \geqslant 0, \quad \forall y \in C_k.$$

Setting $y = x^*$, we obtain

$$\langle v^k - x^k, x^* - v^k\rangle \geqslant -\frac{f(v^k, x^*)}{\gamma_k} \geqslant 0, \quad \forall k \in \mathbb{N}. \tag{4.32}$$

Thus,

$$\begin{aligned}
\|x^k - x^*\|^2 &= \|(x^k - v^k) + (v^k - x^*)\|^2 \\
&= \|x^k - v^k\|^2 + 2\langle x^k - v^k, v^k - x^*\rangle + \|v^k - x^*\|^2 \\
&\geqslant \|x^k - v^k\|^2 + \|v^k - x^*\|^2 \qquad (4.33) \\
&\geqslant \|v^k - x^*\|^2, \quad \forall k \in \mathbb{N}, \qquad (4.34)
\end{aligned}$$

where in the first inequality we used (4.32). Therefore,

$$\begin{aligned}\|x^{k+1} - x^*\| &\leqslant \|x^{k+1} - v^k\| + \|v^k - x^*\| \\ &\leqslant \|x^k - x^*\| + \varepsilon_k, \quad \forall k \in \mathbb{N}, \end{aligned} \tag{4.35}$$

where we used both (4.22) and (4.34) in the second inequality. Thus, $\{x^k\}$ is quasi-Fejér convergent to S^*.

(ii): From Lemma 3, we have that the sequence $\{\|x^k - x^*\|\}$ converges, for all $x^* \in S^*$. Let us denote $\lim_{k\to+\infty} \|x^k - x^*\| = \mu \geqslant 0$. Note that $\lim_{k\to+\infty} \|v^k - x^*\| = \mu$. Indeed, combining (4.34) and (4.35), we have

$$\|x^{k+1} - x^*\| \leqslant \|v^k - x^*\| + \varepsilon_k \leqslant \|x^k - x^*\| + \varepsilon_k.$$

Taking the limit as $k \to +\infty$ in the last inequality and taking into account that $\lim_{k\to\infty} \varepsilon_k = 0$, we prove the claim. Now, from (4.33) and (4.34), we have

$$\|x^k - x^*\|^2 \geqslant \|x^k - v^k\|^2 + \|v^k - x^*\|^2 \geqslant \|v^k - x^*\|^2, \quad \forall k \in \mathbb{N}.$$

Taking the limit as $k \to +\infty$ in the last inequality and taking into account that $\lim_{k\to\infty} \|x^k - x^*\| = \lim_{k\to\infty} \|v^k - x^*\| = \mu$, we prove the second assertion of item (ii). On the other hand,

$$0 \leqslant \|x^{k+1} - x^k\| \leqslant \|x^{k+1} - v^k\| + \|v^k - x^k\| \leqslant \varepsilon_k + \|v^k - x^k\|.$$

Taking the limit as $k \to +\infty$ in the last inequality and taking into account that $\lim_{k\to\infty} \|x^k - v^k\| = 0$ and $\lim_{k\to\infty} \varepsilon_k = 0$, we prove the first assertion of item (ii) and the proof is concluded. □

Next, we show the main convergence result for Algorithm 8.

Theorem 13. *Let $\{x^k\}$ be the sequence generated by Algorithm 8. Then, every weak cluster point of $\{x^k\}$ is a solution to the $\mathsf{QEP}(f, C)$.*

Proof: From Proposition 14 (i), we have that $\{x^k\}$ is quasi-Fejér convergent to S^*. Thus, from Lemma 2, $\{x^k\}$ is bounded. Let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ weakly converging to x^*. Note that from Proposition 14 (ii), we have that $x^{k_j+1} \rightharpoonup x^*$ and $v^{k_j} \rightharpoonup x^*$. Now, from the definition of the algorithm, $v^k \in S_{EP}(f_k, C_k)$ for all $k \in \mathbb{N}$, and, in particular, $v^{k_j} \in C(x^{k_j})$. Therefore, since C is M-continuous, we obtain that $x^* \in C(x^*)$. On the other hand,

$$\begin{aligned} 0 &\leqslant f(v^{k_j}, y) + \gamma_{k_j}\langle v^{k_j} - x^{k_j}, y - v^{k_j}\rangle \\ &\leqslant f(v^{k_j}, y) + \gamma_{k_j}\|v^{k_j} - x^{k_j}\| \, \|y - v^{k_j}\|, \quad \forall y \in C_{k_j}. \end{aligned}$$

Taking the limit as $j \to +\infty$ in last inequality taking into account that the sequences $\{\gamma_k\}$, $\{x^k\}$ and $\{v^k\}$ are bounded, from Proposition 14 (ii) $\lim_{j\to+\infty} \|x^{k_j} - v^{k_j}\| = 0$ and i) in Assumptions 1, we have

$$f(x^*, y) \geqslant 0, \quad \forall y \in C(x^*),$$

having in mind that C is M-continuous. This means that $x^* \in S_{QEP}(f, C)$ and the proof is completed. □

4.2.2 Numerical experiments

In this section, we illustrate the performance of the proposed methods on some test problems. The first and second examples are 2-dimensional and 5-dimensional academic quasi-equilibrium problems, respectively, considered in [47] and the third example is a quasi-variational inequality formulation of a generalized Nash game given by [49]. In each example, we compare the performance of both inexact versions with the exact proximal point method for quasi-equilibrium problems proposed by [47]. Since the main novelty of this paper is to propose inexact versions of the proximal point method and investigate their performance compared with the exact method considered in [47], we refrain from discussing the computational efficiency of other methods, and hence we skip discussions of comparisons of the proposed methods with other methods.

In order to investigate the performance of Algorithm 7 and 8, we consider these algorithms such that all the iterations x^k do not coincide with the corresponding iteration of the proximal point method. For Algorithm 7, this is shown in Figures 4.5, 4.7 and 4.9, where $\|e^k\| \neq 0$, for all k. In Algorithm 8, we consider $x^{k+1} \in B[v^k, \epsilon_k] = \{x \in X : \|x - v^k\| = \epsilon_k\}$, where v^k is the (exact) iteration of the proximal point method and $\epsilon_k = \frac{1}{k^2}$.

The algorithm is coded in MATLAB R2020b on an 8 GB RAM Intel Core i7 to obtain the numerical results. The stopping rule is $\|x^{k+1} - x^k\| < 10^{-5}$. We take $\gamma_k = \gamma$, for all $k \in \mathbb{N}$, and the parameter γ will be specified in each problem. We solve the subproblem by using the regularized method in Muu and Quoc [35]. They consider the following iterative method for solving equilibrium problems as in (2.1): for any starting point $x^0 \in X$ and $\gamma > 0$, given $x^k \in X$ define $x^{k+1} \in X$ such that

$$x^{k+1} = \arg\min_{y\in X}\{\gamma f(x^k, y) + \frac{1}{2}\|y - x^k\|^2\}. \tag{4.36}$$

The solutions of the subproblems in (4.36) are computed by a built -in MATLAB solver: “fmincon" if we deal with a constrained non-linear problem and “quadprog" if the subproblem reduces to a quadratic programming with linear constraints.

Next, we present the examples and the results of the (inexact) Algorithm 7 and 8 comparing their performance with the (exact) Proximal Point Method (PPM) proposed in [47]. To this end, we run the three algorithms with 100 random starting points in the box $[-5, 5]^n \subset \mathbb{R}^n$ for different values of $\gamma > 0$ in the method (4.36) for solving the subproblems. At each run, the methods start from the same initial point and use the same constant γ. In the tables, we show the results of all methods in terms of the number of iterations until the stopping rule is satisfied and the accuracy $\Upsilon(x^k)$ (cf. Proposition 6). In the tables, min. iter., max. iter. and aver. iter. denote the minimal, maximum and average of iterations in 100 runs of the methods, respectively, as well as aver. $\Upsilon(x^k)$ stands for the average of the values $\Upsilon(x^k)$ in 100 runs, where x^k is the solution found by the methods. In the figures, we consider a particular run of each method (using the same x^0 and γ) presenting the accuracy at each step as well as the error sequences of the Algorithm 7.

Example 8. *[47, Example 4.1] Consider the* 2*-dimensional non-smooth quasi-equilibrium problem defined by the bifunction*

$$f(x, y) = \mid y_1 \mid - \mid x_1 \mid + y_2^2 - x_2^2$$

and the multivalued mapping C *given by*

$$C(x) = \left\{ y \in \mathbb{R}^2_+ \, ; \, y_1 + y_2 = 1 + \frac{\mid x_1 \mid}{1 + \mid x_1 \mid} \right\}.$$

One can check that f *is monotone and the solution set is the single point* $x^* = (1, \frac{1}{2})$. *Furthermore, if* $X = \{(x_1, x_2) \in \mathbb{R}^2_+ \, : \, |x_1| = 1$ *and* $x_2 \geqslant \frac{1}{2}\}$, *then* $x^* \in S^*$. *On the other hand, since* $x^* \in C(x^*)$, *we have that* $S_* \neq \varnothing$ *for every* $X \subset \mathbb{R}^2$ *such that* $EP(f, X)$ *has a non-empty solution set* S_{EP}.

In Tables 4.4, 4.5 and 4.6, we can see that Algorithm 7 outperforms PPM and Algorithm 8 in both the number of iterations and the median of the quality of the solution found in all cases considered. On the other hand, Algorithm 8 under-performs PPM in all cases considered. In particular, for $\gamma = 1.5$ (see Table 4.6) the performance of Algorithm 8 and PPM was similar.

Table 4.4: Results for Example 8 with $\gamma = 5.5$.

Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	7	10	8.65	7.9279e-09
Algorithm 8	16	16	16	9.1332e-07
PPM [47]	8	13	11.01	2.3026e-08

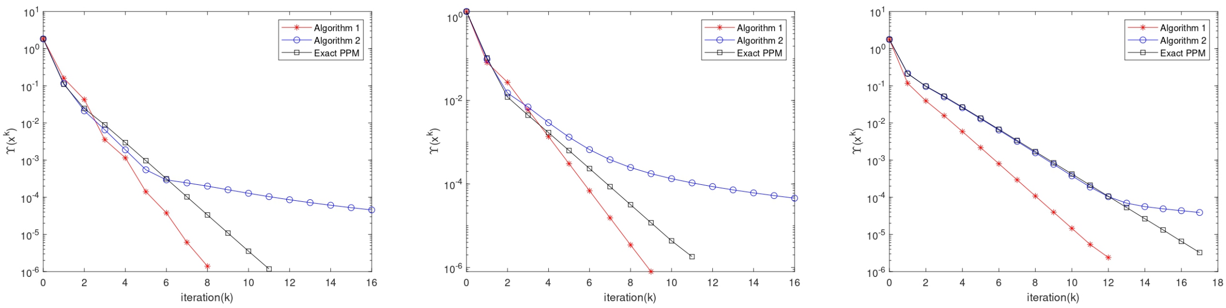
Table 4.5: Results for Example 8 with $\gamma = 3.5$.

Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	6	10	8.75	1.8384e-08
Algorithm 8	15	16	15.74	9.1429e-07
PPM [47]	6	14	12.47	2.4065e-08

Table 4.6: Results for Example 8 with $\gamma = 1.5$.

Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	8	14	12.39	4.4741e-08
Algorithm 8	15	20	17.87	7.2625e-07
PPM [47]	11	20	17.44	8.1751e-08

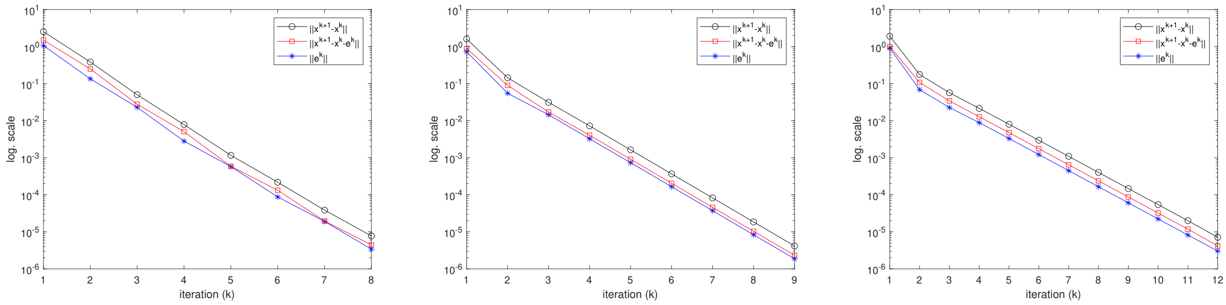
In Figures 4.4 and 4.5, we consider random starting points described in each figure for $\gamma = 5.5$ (in Figures 4.4a and 4.5a), $\gamma = 3.5$ (in Figures 4.4b and 4.5b) and $\gamma = 1.5$ (in Figures 4.4c and 4.5c).



(a) $x^0 = (-1.24, 0.46)$ (b) $x^0 = (0.61, -1.04)$ (c) $x^0 = (-1.01, 0.15)$

Figure 4.4: Accuracy $\Upsilon(x^k)$ in each method (using log. scale) for Ex. 8.

Considering different starting points and values for the constant γ, we can see in Figure 4.4 that Algorithm 7 obtains a similar performance in terms of the accuracy $\Upsilon(x^k)$ to the PPM but in fewer iterations. In this context, Algorithm 8 underperforms Algorithm 7 and PPM in both accuracy and number of iterations. In Figure 4.5, we show that the error $\{e^k\}$ and the sequence generated by Algorithm 7 satisfy the assumptions (E1) and (E2) for Example 8.



(a) $x^0 = (-1.24, 0.46)$ (b) $x^0 = (0.61, -1.04)$ (c) $x^0 = (-1.01, 0.15)$

Figure 4.5: Algorithm 7: error satisfying (E1) and (E2) for Ex. 8.

Example 9. *[47, Example 4.2] Consider the quasi-equilibrium problem where the multi-valued mapping* C *is given by*

$$C(x) = \prod_{1 \leqslant i \leqslant 5} C_i(x),$$

where for each $x \in \mathbb{R}^5$ *and each* i*, the set* $C_i(x)$ *is defined by*

$$C_i(x) = \left\{ y_i \in \mathbb{R} \,;\, y_i + \sum_{1 \leqslant j \leqslant 5,\, j \neq i} x_j \geqslant -1 \right\}$$

and the bifunction f *is of the form*

$$f(x, y) = \langle Px + Qy + q,\, y - x \rangle,$$

in which the matrices P*,* Q *and the vector* q *are given by*

$$P = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}; \quad Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \textit{and} \quad q = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{bmatrix}.$$

Note that this problem, in general, is not a quasi-variational inequality problem. The bifunction f *is monotone and it comes from the Nash-Cournot equilibrium model; for more details see [49] and references therein. Now, let us consider* $X = \{x \in \mathbb{R}^5 : \sum_{i=1}^{5} x_i \geqslant -1; x_i \in [-5, 5]\}$. *One has*

$$x \in C(x) = \prod_{i=1}^{n} C_i(x) \Leftrightarrow x_i \in C_i(x) \Leftrightarrow \sum_{i=1}^{5} x_i \geqslant -1.$$

Therefore, $x \in C(x)$, *for all* $x \in X$. *Moreover, from [42, Section 6], we have that* $x^* = [-0.725, 0.803, 0.719, -0.865, 0.250]$ *satisfies* $f(x^*, y) \geqslant 0$, *for all* $y \in X$. *Thus,* $x^* \in S_*$.

In Tables 4.7, 4.8 and 4.9, we can see that Algorithm 7 outperforms PPM and Algorithm 8 in both the number of iterations and the median of the quality of the solution found in all cases considered. On the other hand, Algorithm 8 underperforms PPM in all cases considered, but for $\gamma = 0.4$ (see Table 4.7) and $\gamma = 0.5$ (see Table 4.8) the performance of Algorithm 8 and PPM was similar.

Table 4.7: Results for Example 9 with $\gamma = 0.4$.

Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	13	18	15.45	1.1330e-07
Algorithm 8	16	20	18.27	9.9584e-07
PPM [47]	16	20	18.08	1.2109e-07

Table 4.8: Results for Example 9 with $\gamma = 0.5$.

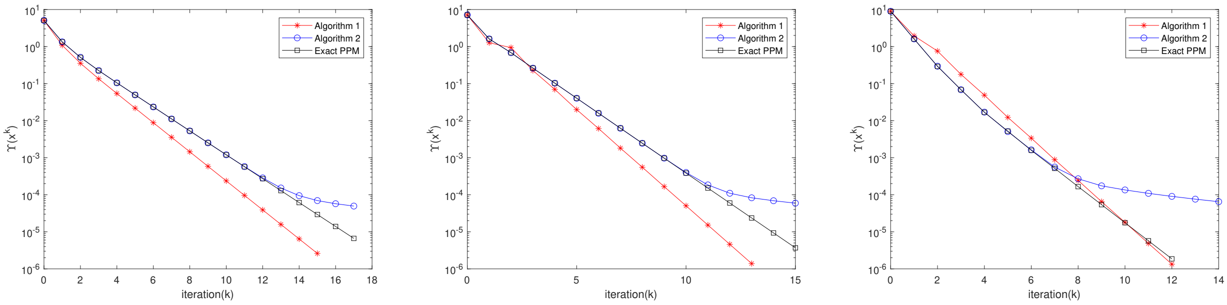
Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	11	16	12.74	6.7091e-08
Algorithm 8	15	18	15.85	1.0383e-06
PPM [47]	12	18	15.20	8.4498e-08

In Figures 4.6 and 4.7, we consider random starting points described in each figure for $\gamma = 0.4$ (in Figures 4.6a and 4.7a), $\gamma = 0.5$ (in Figures 4.6b and 4.7b) and $\gamma = 0.6$ (in Figures 4.6c and 4.7c).

Considering different starting points and values for the constant γ, we can see in Figure 4.6 that Algorithm 7 obtains a similar performance in terms of the accuracy $\Upsilon(x^k)$

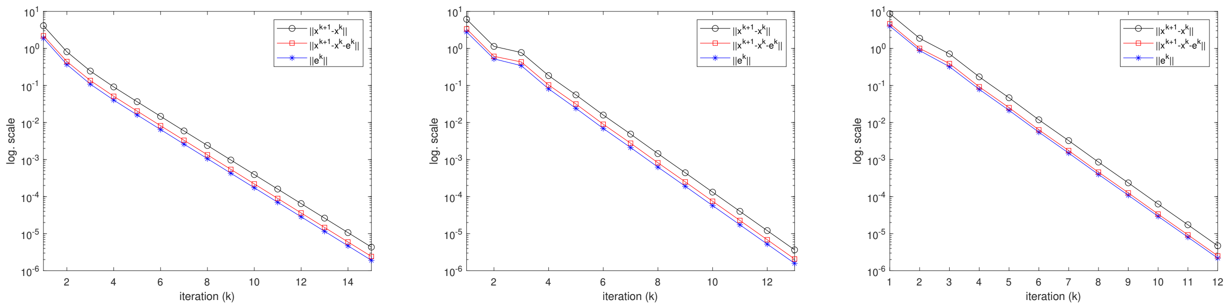
Table 4.9: Results for Example 9 with $\gamma = 0.6$.

Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	10	18	13.03	1.6718e-08
Algorithm 8	14	16	14.68	1.1231e-06
PPM [47]	9	16	13.43	3.1780e-08



(a) $x^0 = (-1.48, 3.30, 0.85, 0.49, 4.17)$ (b) $x^0 = (-3.42, 4.70, 4.57, -0.14, 3.00)$ (c) $x^0 = (-3.58, -0.78, 4.15, 2.92, 4.59)$

Figure 4.6: Accuracy $\Upsilon(x^k)$ in each method (using log. scale) for Ex. 9.



(a) $x^0 = (-1.48, 3.30, 0.85, 0.49, 4.17)$ (b) $x^0 = (-3.42, 4.70, 4.57, -0.14, 3.00)$ (c) $x^0 = (-3.58, -0.78, 4.15, 2.92, 4.59)$

Figure 4.7: Algorithm 7: error satisfying (E1) and (E2) for Ex. 9.

to the PPM but in less iterations except in Figure 4.7c where these two methods have similar performance also in number of iterations when we take $\gamma = 0.6$. As we can see in Table 4.9, in this case ($\gamma = 0.6$) Algorithm 7 slightly wins the PPM in the median of iterations and accuracy. On the other hand, Algorithm 8 underperforms Algorithm 7 and PPM in both accuracy and number of iterations but with similar performance for $\gamma = 0.4$ and $\gamma = 0.5$; see Table 4.7 and 4.8, respectively. In Figure 4.7, we show that the error $\{e^k\}$ and the sequence generated by Algorithm 7 satisfy the assumptions (E1) and (E2) for Example 9.

Example 10. *[49, Example 2] Consider the 2-dimensional quasi-variational inequality problem given by*

$$F(x) = \left(2x_1 + \frac{8}{3}x_2 - 34, 2x_2 + \frac{5}{4}x_1 - 24.25\right)$$

and the multivalued mapping defined by $C(x) = C_1(x_2) \times C_2(x_1)$*, where* $C_1(x_2) = \{y_1 \in \mathbb{R}\,;\, 0 \leqslant y_1 \leqslant 10,\, y_1 \leqslant 15 - x_2\}$ *and* $C_2(x_1) = \{y_2 \in \mathbb{R}\,;\, 0 \leqslant y_2 \leqslant 10,\, y_2 \leqslant 15 - x_1\}$*. Its solution set is the point* $(5, 9)$ *and the line segment* $[(9, 6), (10, 5)]$*. One can check that the equilibrium bifunction is monotone. Furthermore, if* $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leqslant x_1 \leqslant 6$ *and* $0 \leqslant x_2 \leqslant 10\}$ *and* $x^* = (5, 9)$*, then we can prove that* $x^* \in S^*$ *and* $x^* \in S_*$*.*

In Tables 4.10, 4.11 and 4.12, we can see that Algorithm 7 outperforms PPM and Algorithm 8 both in number of iterations and median of the quality of the solution found in all cases considered. In this example, Algorithm 8 and PPM have similar performance in all cases considered in both number of iterations and accuracy. These facts are clearly shown in Figure 4.8. In Figure 4.9, we show that the error $\{e^k\}$ and the sequence generated by Algorithm 7 satisfy the assumptions (E1) and (E2) for Example 10.

Table 4.10: Results for Example 10 with $\gamma = 0.15$.

Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	244	288	251.86	8.3892e-07
Algorithm 8	351	376	358.70	1.2894e-06
PPM [47]	352	377	359.45	1.2872e-06

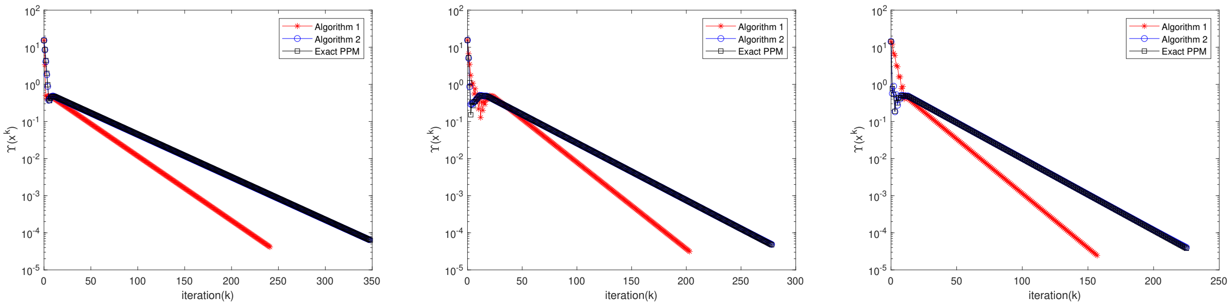
Table 4.11: Results for Example 10 with $\gamma = 0.2$.

Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	195	212	203.35	6.2295e-07
Algorithm 8	271	295	277.84	9.9035e-07
PPM [47]	271	298	278.34	9.3600e-07

In Figures 4.8 and 4.9, we consider random starting points described in each figure for $\gamma = 0.15$ (in Figures 4.8a and 4.9a), $\gamma = 0.2$ (in Figures 4.8b and 4.9b) and $\gamma = 0.25$ (in Figures 4.8c and 4.9c).

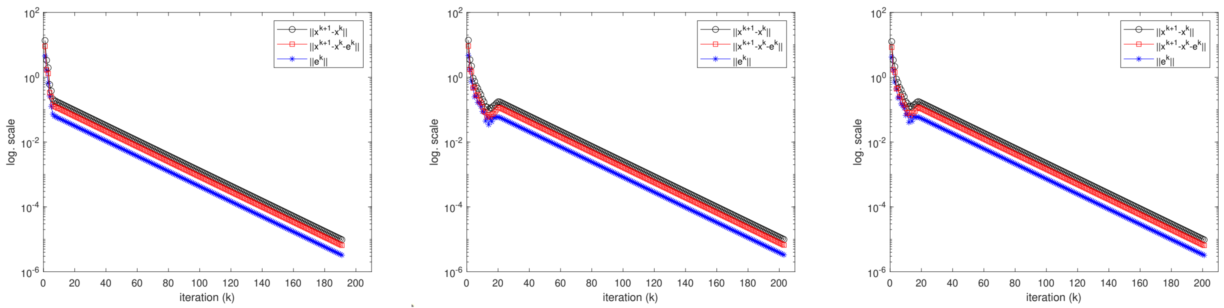
Table 4.12: Results for Example 10 with $\gamma = 0.25$.

Algorithm	min. iter. (k)	max. iter. (k)	aver. iter. (k)	aver. $\Upsilon(x^k)$
Algorithm 7	157	160	158.59	4.9615e-07
Algorithm 8	222	255	229.63	7.7335e-07
PPM [47]	222	252	229.49	7.3511e-07



(a) $x^0 = (-1.80, 0.21)$ (b) $x^0 = (-1.24, -0.71)$ (c) $x^0 = (-0.38, 0.19)$

Figure 4.8: Accuracy $\Upsilon(x^k)$ in each method (using log. scale) for Ex. 10.



(a) $x^0 = (-1.80, 0.21)$ (b) $x^0 = (-1.24, -0.71)$ (c) $x^0 = (-0.38, 0.19)$

Figure 4.9: Algorithm 7: error satisfying (E1) and (E2) for Ex. 10.

Chapter 5

Aplication to the Cournot model

The Cournot model of a duopoly is at the origin of game theory; see Cournot [9]. Usually, in textbooks, it has been modelled as an equilibrium problem; see Osborne [37]. Let us see that we can improve its presentation as a quasi-equilibrium problem through its presentation in mathematics as a Nash Cournot game with a unique shared constraint. This point of view is well known; see [11]. But almost no effort has been made to show this in an intuitive way in the context of the Cournot model of a duopoly. The drawback is that most of the intuitions are lost.

5.1 The viability condition of the Cournot model as an EP

We recall the celebrated Cournot model of a duopoly. It considers two producers $j = 1, 2$ of an homogeneous good that produce the quantities $x = (x^1, x^2) \in \mathbb{R}^2_+$ of this good at the unit costs $(c^1, c^2) \in \mathbb{R}^2_{++}$. In this usual presentation, we will suppose that unit costs of production are the same: $c^1 = c^2 = c > 0$.

The two producers sell this good at the same unit price

$$p(Q) = \begin{cases} a - Q \in \mathbb{R}, \text{ if } 0 \leqslant Q \leqslant a \\ 0, \text{ if } Q > a, \end{cases}$$

where $a > 0$ is the highest price at which one unit of this good can be sold when Q is close to zero. This price depends on the total level of production of the duopoly, $Q = x^1 + x^2 \in \mathbb{R}_+$. This formula tells us that the more the producers want to produce and sell (Q high), the lower the price $p(Q)$ will be. Think about the oil cartel and the price of oil. The existence of this duopoly requires that the two producers do not produce too much in total: $p(Q) = a - Q \in \mathbb{R}_+$, if $0 \leqslant Q \leqslant a$. If not, the price of the good will go to zero: $p(Q) = 0$, if $Q > a$. Then, the duopoly breaks (selling at a zero or negative price is not rewarding).

Let us show that a Cournot equilibrium x^* verifies the two QEP conditions : $x^* \in C(x^*)$ and $f(x^*, y) \geqslant 0$, for all $y \in C(x^*)$.

5.2 Setting the fixed point condition of a QEP as a viability condition

The viability constraint of the duopoly is that the price of the final good will be non-negative, i.e., $p(Q) \in \mathbb{R}_+$, which means, $p(x) = a - (x^1 + x^2) \geqslant 0$. That is, $x = (x^1, x^2) \in K$, where

$$K = \left\{x = (x^1, x^2) \in \mathbb{R}^2_+, \quad a - (x^1 + x^2) \geqslant 0 \right\}$$

is the two dimensional simplex if $a = 1$.

Denote by $C^j(x^{-j}) = \left\{x^j \in \mathbb{R}_+,\ 0 \leqslant x^j \leqslant a - x^{-j}\right\}$, $j = 1, 2$ and $C(x) = C^1(x^2) \times C^2(x^1)$. It is easy to show that $x \in C(x)$ is equivalent to $x^j \in C^j(x^{-j})$, $j = 1, 2$, i.e., $x \in C(x)$, and hence, it is equivalent to $x = (x^1, x^2) \in K$. The term shared constraint comes from the fact of a unique constraint K (i.e., a non-negative selling price) defines all the moving sets $C^j(x^{-j})$.

5.3 Setting the equilibrium condition of a QEP

The margins of duopolists are the difference $m^i = p - c^i$, $i = 1, 2$, between the price of the good and their unit costs of production and selling with $c^1 = c^2 = c > 0$. Then, at the status quo $x = (x^1, x^2) \in \mathbb{R}^2_+$ their profits are, for $i = 1, 2$,

$$\pi^i = \begin{cases} px^i - cx^i = (p - c)x^i = [(a - c - x^1 - x^2]\, x^i, \text{ if } x^1 + x^2 \leqslant a, \\ -cx^i, \text{ if } x^1 + x^2 > a. \end{cases}$$

Suppose now that, in order to maximize his profit, each duopolist i considers changing unilaterally his production from the status quo. This means that he hopes that the other producer will not change, producing the same level as before. That is, each duopolist maximizes his profit, taking as given what the other duopolist produces and sells at the status quo $x = (x^1, x^2) \in \mathbb{R}^2_+$. Then, if the price of the good is non-negative, i.e., if the viability constraint $p = a - (x^1 + x^2) \geqslant 0$ is satisfied, they solve the unconstraint interrelated programs

$$\max \left\{ \pi^1(y^1, x^2) = \left[a - c - x^2 - y^1\right] y^1, y^1 \in X^1 = \mathbb{R}_+ \right\}$$

and

$$\max \left\{ \pi^2(x^1, y^2) = \left[a - c - x^1 - y^2\right] y^2, y^2 \in X^2 = \mathbb{R}_+ \right\}.$$

Their resolutions provide the best responses $x^{*,1} = \frac{1}{2}\left[a - c^1 - x^2\right]$ and $x^{*,2} = \frac{1}{2}\left[a - c^2 - x^1\right]$. Then, the Cournot equilibrium is a fixed point $x^* = (x^{*,1}, x^{*,2}) \in \mathbb{R}^2_+$ such that $x^{*,1} = \frac{1}{2}\left[a - c^1 - x^{*,2}\right]$ and $x^{*,2} = \frac{1}{2}\left[a - c^2 - x^{*,1}\right]$. This gives, if $a > c$,

$$x^* = (x^{*,1}, x^{*,2}) = \frac{1}{3}\left(a - c, a - c\right).$$

In turn, we can verify that $x^* \in K$. Notice that the definition of a constrained Nash equilibrium, with the viability constraint K means that the two following equilibrium conditions are satisfied:

$$\pi^1(x^{*,1}, x^{*,2}) \geqslant \pi^1(y^1, x^{*,2}), \quad \forall\, y^1 \in C^1(x^{*,2}) \tag{5.1}$$

$$\pi^2(x^{*,1}, x^{*,2}) \geqslant \pi^2(x^{*,1}, y^2), \quad \forall\, y^2 \in C^2(x^{*,1}), \tag{5.2}$$

where

$$C^1(x^{*,2}) = \left\{ y^1 \in \mathbb{R}_+,\ a - c - y^1 - x^{*,2} \geqslant 0 \right\}$$

and

$$C^2(x^{*,1}) = \left\{ y^2 \in \mathbb{R}_+,\ a - c - x^{*,1} - y^2 \geqslant 0 \right\}.$$

Then, the equilibrium condition of the QEP $f(x^*, y) \geqslant 0$, for all $y \in C(x^*)$ follows from the definition of a Nikaido-Isoda equilibrium function,

$$f(x, y) = \left[\pi^1(x^1, x^2) - \pi^1(y^1, x^2)\right] + \left[\pi^2(x^1, x^2) - \pi^2(x^1, y^2)\right]$$

with the conditions (5.1) and (5.2).

5.4 Application to the Cournot model of a duopoly

5.4.1 A better formulation of the Nash-Cournot model as a QEP

In this section, we want to:

- remind the usual presentation of the Cournot model of a duopoly as a QEP problem with a unique shared constraint;
- show again that the Cournot example can be better modelled as a quasi equilibrium problem than as an equilibrium problem;
- interpret, in this Cournot context, the linear and inexact regularization method as a perturbation of the unit costs of the duopolists that can lead to a possibly better convergence (at least in some simulations). This is a striking result because it allows, each period, an inexact knowledge of the unit cost that becomes better and better as time evolves if the error is controlled.

Given the new presentation of the Cournot model as a QEP formulated in Section 5, with the same unit costs of production $c^1 = c^2 = c > 0$, the new presentation of the Cournot model as a QEP with different units costs $(c^1, c^2) > 0$, $c^1 \neq c^2$ comes easily if we remind the definition of the selling price $p(Q)$,

$$p(Q) = \begin{cases} a - Q \in \mathbb{R}, \text{ if } 0 \leqslant Q \leqslant a, \\ 0, \text{ if } Q > a. \end{cases}$$

When the unit costs of production of the duopolists are not the same, the margins of the duopolists are not the same. They represent the difference $m^i = p - c^i, i = 1, 2$ between the price of the good and their unit costs of production and selling. Then, the profits of the duopolists are

$\pi^i(x^i, x^{-i}) = px^i - c^i x^i = (p - c^i)x^i = m^i x^i$, $i = 1, 2$. Thus, their profits are,

$$\pi^1 = \begin{cases} [(a - c^1 - x^2) - x^1]\, x^1, \text{ if } x^1 + x^2 \leqslant a, \\ -c^1 x^1, \text{ if } x^1 + x^2 > a, \end{cases}$$

and

$$\pi^2 = \begin{cases} [(a - c^2 - x^1) - x^2]\, x^2, \text{ if } x^1 + x^2 \leqslant a, \\ -c^2 x^2, \text{ if } x^1 + x^2 > a. \end{cases}$$

5.4.2 New viability conditions

As seen in Section 5, the first viability condition for the existence of a duopoly is that the selling price $p = a - (x^1 + x^2) \geqslant 0$ is non-negative. Two new viability conditions that impply this first one require that margins must be non-negative, i.e.,

$$m^1 = \alpha^1(x^2) - x^1 \geqslant 0 \quad \text{and} \quad m^2 = \alpha^2(x^1) - x^2 \geqslant 0,$$

with $\alpha^1(x^2) = a - c^1 - x^2$ and $\alpha^2(x^1) = a - c^2 - x^1$. If not, all profits would be negative.

As previously, each duopolist maximizes his profit, taking as given what the other duopolist produces at the status quo $x = (x^1, x^2) \in \mathbb{R}^2_+$. Then, they solve the unconstraint interrelated programs

$$\max \left\{\pi^1(y^1, x^2) = \left[\alpha^1(x^2) - y^1\right] y^1, y^1 \in X^1 = \mathbb{R}_+\right\}$$

and

$$\max \left\{\pi^2(x^1, y^2) = \left[\alpha^2(x^1) - y^2\right] y^2, y^2 \in X^2 = \mathbb{R}_+\right\}.$$

Their resolutions provide the unique equilibrium

$$x^* = (x^{*,1}, x^{*,2}) = \frac{1}{3}\left(a + c^1 - 2c^2, a + c^2 - 2c^1\right)$$

with $a + c^1 - 2c^2 > 0$ and $a + c^2 - 2c^1 > 0$. If $c^1 = c^2 = c > 0$, then $x^{*,1} = x^{*,2} = \frac{1}{3}(a - c)$.

5.5 The Cournot model as a QEP

A QEP presentation

Set of profitable deviations (moves). Two viability constraints are that the profit of each duopolist must not be negative when they try to move unilaterally from the status quo profile of productions $x = (x^1, x^2) \in \mathbb{R}^2_+$ to another profile of production $y = (y^1, y^2) \in \mathbb{R}^2_+$. These constraints are, $y^1 \in C^1(x^2)$ and $y^2 \in C^2(x^1)$ where,

$$C^1(x^2) = \left\{y^1 \in \mathbb{R}_+, \pi^1(y^1, x^2) = \left[\alpha^1(x^2) - y^1\right] y^1 \geqslant 0\right\}$$

and

$$C^2(x^1) = \left\{y^2 \in \mathbb{R}_+, \pi^2(x^1, y^2) = \left[\alpha^2(x^1) - y^2\right] y^2 \geqslant 0\right\}.$$

These two sets represent the sets of unilateral profitable deviations (y^1, x^2) and (x^1, y^2) from the status quo x. Then, these two viability (profitability) constraints can be written

$$y^1 \in C^1(x^2) = \left\{y^1, 0 \leqslant y^1 \leqslant \alpha^1(x^2)\right\}$$

and

$$y^2 \in C^2(x^1) = \left\{y^2, 0 \leqslant y^2 \leqslant \alpha^2(x^1)\right\}.$$

That is, $y \in C(x)$ if $y = (y^1, y^2)$ and $C(x) = C^1(x^2) \times C^2(x^1)$. More explicitly, $y \in C(x)$ is equivalent to $0 \leqslant y^1 \leqslant \alpha^1(x^2) = a - c^1 - x^2$ and $0 \leqslant y^2 \leqslant \alpha^2(x^1) = a - c^2 - x^1$. If $c^1 = c^2 = c > 0$, then $y \in C(x)$ is equivalent to $0 \leqslant y^1 \leqslant a - c - x^2$ and $0 \leqslant y^2 \leqslant a - c - x^1$.
Several viability constraints: making non-negative profits at the status quo. As a consequence, the condition $x^* \in C(x^*)$ means that it can be viable (profitable) to stay at the status quo, because each duopolist makes a non-negative profit. More explicitly, $x^* \in C(x^*)$ is equivalent to $0 \leqslant x^{*,1} \leqslant \alpha^1(x^{*,2}) = a - c^1 - x^{*,2}$ and $0 \leqslant x^{*,2} \leqslant \alpha^2(x^{*,1}) = a - c^2 - x^{*,1}$, i.e., $\pi^1(x^{*,1}, x^{*,2}) \geqslant 0$ and $\pi^2(x^{*,1}, x^{*,2}) \geqslant 0$. This refers to the existence of two viability conditions instead of one when $c^1 = c^2 = c > 0$.
Equilibrium: stability of the status quo with different moving constraints. The status quo $x^* = (x^{*1}, x^{*,2})$ is an equilibrium if it is not profitable to deviate unilaterally from it, i.e., iff,

$$\pi^1(y^1, x^{*,2}) \leqslant \pi^1(x^{*1}, x^{*,2}), \quad \forall y^1 \in C^1(x^{*,2})$$

and

$$\pi^2(x^{*,1}, y^2) \leqslant \pi^2(x^{*,1}, x^{*2}), \quad \forall y^2 \in C^2(x^{*,1}).$$

A better QEP formulation of the Cournot equilibrium problem. It becomes: find $x^* \in C(x^*)$ such that $f(x^*, y) \geqslant 0$, for all $y \in C(x^*)$ with a Nikaido–Isoda function

$$f(x, y) = \left[\pi^1(x^1, x^2) - \pi^1(y^1, x^2)\right] + \left[\pi^2(x^1, x^2) - \pi^2(x^1, y^2)\right].$$

5.6 The advantages of a QEP formulation

A richer concept of equilibria with viability constraints

A richer concept of the status quo as an equilibrium emerges. It takes care of the fact that

(i) if you stay, it is viable to stay because each duopolist makes a non-negative profit. This is a viability constraint that the traditional presentation ignores. See above the usual presentation;

(ii) if you change, a change will not be profitable for each duopolist if the other duopolist does not change. This is a stability constraint. The traditional presentation uses it, but it fails to take care of the local and moving constraints $y^1 \in C^1(x^{*,2})$; and $y^2 \in C^2(x^{*,1})$. This highlights the important idea that if you think that what the other will do will constraint your choice set, you must take care of this fact. The traditional presentation misses this point.

A better understanding of the generalized Nash equilibrium problem

Existence of QEP. Particular cases of quasi-equilibrium problems are generalized Nash equilibrium problems; see, for instance, Dutang [11] and Fischer et al. [15] for games with shared constraints. Our simple and new Cournot perspective helps to better understand the intuitions behind sufficient conditions given for the existence of equilibria for games with shared constraints.

A linear and inexact regularization of a QEP leads to a striking interpretation of the Cournot model

Linear regularization. Our work propose the linear regularization (4.20) of the "loss to deviate from the status quo" function $f(x, y)$. In the Cournot example

$$f(x,y) = \left[\pi^1(x^1,x^2) - \pi^1(y^1,x^2)\right] + \left[\pi^2(x^1,x^2) - \pi^2(x^1,y^2)\right]$$

and

$$\langle e, y - x\rangle = e^1(y^1 - x^1) + e^2(y^2 - x^2).$$

Then, for duopolist 1,

$$\pi^1(y^1,x^2) = (a - c^1 - x^2)y^1 - (y^1)^2$$

and

$$\pi^1(x^1,x^2) = (a - c^1 - x^2)x^1 - (x^1)^2,$$

give $\pi^1(y^1,x^2) - \pi^1(x^1,x^2) = (a - c^1 - x^2)(y^1 - x^1) - [(y^1)^2 - (x^1)^2]$. Then,

$$\pi^1(y^1,x^2) - \pi^1(x^1,x^2) - e^1(y^1 - x^1) = \left[a - (c^1 + e^1) - x^2\right](y^1 - x^1) - \left[(y^1)^2 - (x^1)^2\right].$$

The same for duopolist 2:

$$\pi^2(x^1, y^2) - \pi^2(x^1, x^2) - e^2(y^2 - x^2) = \left[a - (c^2 + e^2) - x^1\right](y^2 - x^2) - \left[(y^2)^2 - (x^2)^2\right].$$

The QEP presentation of the Cournot model becomes: find $x^* \in X$ such that $x^* \in C(x^*)$ with $f(x^*, y) \geqslant 0$ for all $y \in C(x^*)$.

A striking interpretation. The previous calculations show that a linear and inexact perturbation represents a perturbation of the unit cost of each duopolist. This is a marvellous result in behavioral sciences. Indeed, our work shows that a linear and inexact perturbation allows:

(i) to a better computational performance than the exact one;

(ii) to the inexact knowledge of the unit costs which can become better and better as time evolves a controlled error.

Chapter 6

Conclusion

In this work, we propose three variations of the proximal point method in the context of quasi-equilibrium. In the first one, we propose an exact algorithm that makes use of the Bregman distance in $\mathbb{R}^n$ space. We study its convergence analysis, obtaining some classic results derived from the proximal point method. To a test problem, we compare its performance, illustrating the superiority of Bregman functions over the Euclidean norm.

For the inexact versions, we consider two variations of the proximal point method designed to solve quasi-equilibrium problems in Hilbert spaces. In one of these variations, we consider an approximate regularized bifunction, while in the other, we take a point within a neighborhood of the exact solution. We provide convergence analyses for these methods under standard assumptions and illustrate their performance relative to the exact method.

Moreover, we propose an original and intuitive interpretation of the Cournot duopoly model as a quasi-equilibrium problem, which is more rigorous than its formulation as an equilibrium problem.

Regarding future work, our plan is to continue our research on quasi-equilibrium problems. We aim to explore works that rely on more relaxed assumptions compared to those used in this study. Furthermore, we aim to continue our study of extending algorithms from equilibrium problems to the quasi-equilibrium context.

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